

March 30th - April 4th, Trento, Italy

11th International Conference on

**High-Order Nonlinear Numerical Methods for
Evolutionary PDEs: Theory and Applications**

BOOK OF ABSTRACTS

Organized by the University of Trento

The instrument that mediates between theory and practice, between thought and observation, is mathematics; it builds the connecting bridge and makes it stronger and stronger. Thus it happens that our entire present-day culture, insofar as it rests on intellectual insight into and harnessing of nature, is founded on mathematics. Already, Galileo said: Only he can understand nature who has learned the language and signs by which it speaks to us; but this language is mathematics and its signs are mathematical figures. Kant declared, "I maintain that in each particular natural science there is only as much true science as there is mathematics." In fact, we do not master a theory in natural science until we have extracted its mathematical kernel and laid it completely bare. Without mathematics today's astronomy and physics would be impossible; in their theoretical parts, these sciences unfold directly into mathematics. These, like numerous other applications, give mathematics whatever authority it enjoys with the general public.

DAVID HILBERT

Radio address given in Königsberg on 8 September 1930, on the occasion of the yearly meeting of the Society of German Natural Scientists and Physicians (English translation by James T. Smith)

This conference is dedicated to

the 80th birthday of Prof. Dr. Dr. h.c. Eleuterio F. Toro, OBE

the 70th birthday of Prof. Christian Klingenberg

the 65th birthday of Prof. Rémi Abgrall

the 60th birthday of Prof. Lorenzo Pareschi

the 60th birthday of Prof. Pep Mulet

the 60th birthday of Prof. Angelo Iollo

the 60th birthday of Prof. Elena Vázquez Cendón

the 50th birthday of Prof. Mario Ricchiuto

the 50th birthday of Prof. Vladimir Titarev

the 40th birthday of Prof. Walter Boscheri

Scientific committee:

Rémi Abgrall (*University of Zurich, Switzerland*)

Michael Dumbser (*University of Trento, Italy*)

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This conference has been supported by:

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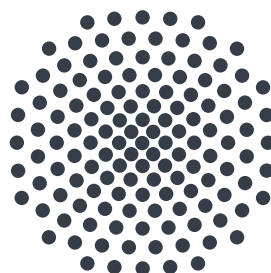
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Welcome to HONOM 2026!

We are very pleased to welcome you to HONOM 2026, *the 11th International Conference on High Order Nonlinear Numerical Methods for Evolutionary PDE, held in Trento, Italy.*

HONOM 2026 is the eleventh in a series of biannual conferences that began with a meeting in Trento, Italy in 2005. Further editions of this conference were held in Trento, Italy in the years 2007, 2009, 2011, Bordeaux, France in 2013, Trento, Italy in 2015, Stuttgart, Germany in 2017, Madrid, Spain in 2019, Braga, Portugal in 2022 and Chania, Greece in 2024. These conferences focus on recent developments and new directions in the area of numerical methods for nonlinear hyperbolic partial differential equations (PDE).

Many processes in science and engineering can be described by nonlinear time-dependent PDE systems. Accurate and robust numerical methods are required for their solution, since these PDE can exhibit smooth, small and large-scale features as well as discontinuities and singularities at the same time. This is evident in aeroacoustics, gas dynamics, magnetohydrodynamics (MHD), compressible turbulence and general relativity, when attempting to find solutions that may include shocks and singularities, and at the same time one needs to resolve small turbulent eddies and evolve weak signals of large characteristic wavelength over long distances and for large times on relatively coarse grids. In addition to the classical requirement of conservation, a high order of accuracy in both space and time is thus of fundamental importance to properly resolve all physical processes involved. However, since the times of Godunov, it is very well-known that high-order linear schemes generate spurious oscillations (the Gibbs phenomenon). The real challenge, therefore, still remains to construct high-order accurate and at the same time robust and stable nonlinear and non-oscillatory schemes. Significant advances have been made in this direction in the last decades, pioneered by TVD (Total Variation Diminishing) methods, ENO and WENO schemes, as well as nonlinear DG finite element methods with appropriate limiters.

We would like to thank the scientific committee, our sponsors and all the participants for their enriching contributions and we wish you a very pleasant stay in Trento and productive scientific and personal interactions during the conference!

Michael Dumbser

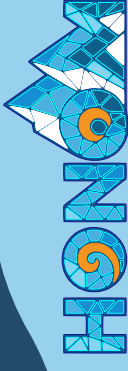
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HONOM 2026 local organizing committee

Daily program

HONOM 2026, 30 March – 3 April 2026, Trento, Italy



Program of the 11th Conference on High Order Nonlinear Numerical Methods for Evolutionary PDE

	Monday, 30.3.2026		Tuesday, 31.3.2026		Wednesday, 1.4.2026		Thursday, 2.4.2026		Friday, 3.4.2026	
	Room Madrizzo	Room Belvedere	Room Madrizzo	Room Belvedere	Room Madrizzo	Room Belvedere	Room Madrizzo	Room Belvedere	Room Madrizzo	Room Belvedere
08:45 – 09:15		Opening and Welcome								
09:15 – 10:00	Droniou		May	Maire	Perrier	Toro				
10:00 – 10:30	Abgrall		Klingenberg	Mulet	Titarev	Boscheri				
10:30 – 11:30	Coffee Break		Coffee Break	Coffee Break	Coffee Break	Coffee Break				
11:30 – 12:00	Ricchiuto		Pareschi	Lukačová	Iollo	Romenski				
12:00 – 12:30	Kurganov		Kummer	Hoch	Puppo	Vilar				
12:30 – 13:00	Frolikovic		Kozubskaya	Gaburro	Thomann	Müller				
13:00 – 15:00	Lunch		Lunch	Lunch	Lunch	Lunch				
15:00 – 15:30	Del Río	Vedral	Dhauadi	Keim	Lampert	Chiocchetti	Willems	Zeravy	Tavelli	Patela
15:30 – 16:00	Lucca	Döhning	Sauerborn	Schwarz	Barthwal	Klima	Preda	Zappa	Peshkov	Artiano
16:00 – 16:30	Zampa	Martínez-Donato	Birke	Kopper	Zanotti	Regener	Anguill	Stuhlmacher	Dumbser	
16:30 – 17:15	Coffee Break		Coffee Break		Coffee Break		Coffee Break		Closure and Coffee Break	
17:15 – 17:45	Dumett	Liu	Munoz Moncayo	Ferrari			Yuan	Careaga		
17:45 – 18:15	Castillo	Samantaray	Barrios	Rinaldi			Helie	Forte Tenreiro		
18:15 – 18:45	Dubois	Redle					Lazareva	Yashina		
20:00 – ...	Welcome Reception		Conference Dinner							

Daily program: Monday, 30 March 2026

08:45 – 09:15 OPENING AND WELCOME – ROOM MADRUZZO

09:15 – 10:00 **Jérôme Droniou** (University of Montpellier)
A polytopal exterior calculus framework with application to the Maxwell equations on manifolds

10:00 – 10:30 **Rémi Abgrall** (University of Zürich)
Recent development on Active flux methodology

10:30 – 11:30 COFFEE BREAK

11:30 – 12:00 **Mario Ricchiuto** (INRIA, Bordeaux)
Genuinely multidimensional Riemann solvers, residual distribution and continuous finite elements: a unified view with some recent results

12:30 – 13:00 **Alexander Kurganov** (Southern University of Science and Technology, Shenzhen)
Local Characteristic Decomposition-Based Central-Upwind Flux Splitting

12:00 – 12:30 **Peter Frolkovic** (Slovak University of Technology, Bratislava)
Implicit methods for numerical solution of hyperbolic systems given by transport equations with nonlinear sorption

13:00 – 15:00 LUNCH

ROOM MADRUZZO

ROOM BELVEDERE

15:00 – 15:30 **Laura Del Río** (INRIA, Bordeaux)
On the heat GLM system and its compatible discretization

Joshua Vedral (TU Dortmund)
An Element-Based Convex Limiting Framework for Continuous Galerkin Methods with Nonlinear Stabilization

15:30 – 16:00 **Alessia Lucca** (University of Trento)
Structure-preserving schemes for symmetric hyperbolic and thermodynamically compatible systems of PDEs

Daniel Döhning (RWTH Aachen University)
Volume Term Adaptivity for Discontinuous Galerkin Schemes

16:00 – 16:30 **Enrico Zampa** (University of Vienna)
A structure preserving finite element scheme for the incompressible GPR model

Izarne Martínez-Donato (University of Valencia)
High-order Minimally Implicit Runge-Kutta (MIRK) methods and application to the 1D shallow water equations with topography and Manning friction

16:30 – 17:15 COFFEE BREAK

17:15 – 17:45 **Miguel Dumett** (San Diego State University)
High Order Mimetic Differences Curl Vector Calculus Identities

Yongle Liu (University of Zürich)
Positivity-preserving Well-balanced PAMPA Scheme with Global Flux Quadrature for One-dimensional Shallow Water Equations

17:45 – 18:15 **Jose Castillo** (San Diego State University)
Data-Driven Mimetic Framework for Turbulence Modeling

Saurav Samantary (Johannes Gutenberg University Mainz)
Asymptotic Preserving and Well Balanced Linearly Implicit Additive IMEX Schemes for the Isentropic Euler Equations with Gravity

18:15 – 18:45 **Juliette Dubois** (TU Berlin)
Uncertainty quantification of conservation laws with several random parameters

Michael Redle (TU Clausthal)
High-Order Positivity-Preserving Well-Balanced Schemes: Combining Flux Globalization with Modified Patankar Methods

20:00 – ... WELCOME RECEPTION

Daily program: Tuesday, 31 March 2026

09:15 **OPENING – ROOM MADRUZZO**

09:15 – 10:00 **Sandra May** (TU Berlin)
About the DoD Stabilization to Solve the Small Cell Problem

10:00 – 10:30 **Christian Klingenberg** (Würzburg University)
A structure-preserving compact high-order method for multi-dimensional hyperbolic conservation laws

10:30 – 11:30 **COFFEE BREAK**

11:30 – 12:00 **Lorenzo Pareschi** (Heriot Watt University, Edinburgh)
Boltzmann Across Regimes: Dynamic Domain Decomposition and Uniformly Accurate IMEX Solvers

12:30 – 13:00 **Florian Kummer** (TU Darmstadt)
Implicit high-order shock tracking for the compressible Euler equations using an extended discontinuous Galerkin method

11:30 – 12:00 **Tatiana Kozubskaya** (Keldysh Institute of Applied Mathematics RAS, Moscow)
EBR Schemes: Recent Developments

13:00 – 15:00 **LUNCH**

ROOM MADRUZZO

ROOM BELVEDERE

15:00 – 15:30 **Firas Dhaouadi** (University of Bordeaux)
On hyperbolic approximations for a class of dispersive and diffusive-dispersive equations

Jens Keim (University of Stuttgart)
An Entropy Stable Discontinuous Galerkin Spectral Element Method on Heterogeneous Grids

15:30 – 16:00 **Robert Sauerborn** (TU Darmstadt)
A Structure Preserving Finite Volume Scheme for the Navier-Stokes-Korteweg Equations

Anna Schwarz (University of Stuttgart)
High-Order Shock Capturing on Heterogeneous Grids using Discontinuous Galerkin Spectral Element Methods

16:00 – 16:30 **Gunnar Birke** (University of Münster)
Energy preservation properties of the Domain of Dependence stabilization method on cut cell meshes

Patrick Kopper (University of Stuttgart)
A p -adaptive discontinuous Galerkin spectral element method on heterogeneous, curvilinear meshes

16:30 – 17:15 **COFFEE BREAK**

17:15 – 17:45 **Carlos Munoz-Moncayo** (KAUST, Saudi Arabia)
A Hyperbolic Approximation of the Nonlinear Schrödinger Equation

Davide Ferrari (INRIA, Strasbourg)
A discrete equivalence in HTC schemes for multiphase flows

17:45 – 18:15 **Gabriel Barrios** (KAUST, Saudi Arabia)
Design of High-Order Fractional-Step Runge-Kutta Methods with Low-Order Subintegrators

Sara Rinaldi (University of Trento)
Arbitrary-Lagrangian-Eulerian solver for relativistic detonation waves

Daily program: Wednesday, 1 April 2026

09:15 **OPENING – ROOM MADRUZZO**

09:15 – 10:00 **Pierre-Henri Maire** (CEA, Bordeaux)
Revisiting Genuinely Multidimensional Fluxes for Cell-Centered Lagrangian Gas Dynamics

10:00 – 10:30 **Pep Mulet** (University of València)
All Mach number second-order implicit-explicit schemes for isentropic compressible Cahn-Hilliard-Navier-Stokes equations

10:30 – 11:30 **COFFEE BREAK**

11:30 – 12:00 **Maria Lukáčová-Medvid'ová** (Johannes Gutenberg University Mainz)
High-Order Methods for Euler Equations: Truly Multidimensional Operators and Selection Criteria

12:00 – 12:30 **Philippe Hoch** (CEA, ArpaJon)
Novelties on arbitrary high-order two-dimensional composite finite volume schemes

12:30 – 13:00 **Elena Gaburro** (University of Verona)
High order accurate ADER Finite Volume schemes based on novel genuinely multidimensional Riemann solvers

13:00 – 15:00 **LUNCH**

ROOM MADRUZZO

15:00 – 15:30 **Joshua Lampert** (University of Hamburg)
Advances in the Construction of Summation-By-Parts Operators on Arbitrary Grids and General Function Spaces

15:30 – 16:00 **Rahul Barthwal** (University of Stuttgart)
A second order accurate generalized Riemann solver for a hyperbolic model of two-layer thin film flow

16:00 – 16:30 **Olindo Zanotti** (University of Trento)
High order numerical schemes within a new first-order BSS-NOK formulation of the Einstein equations

16:30 – 17:15 **COFFEE BREAK**

ROOM BELVEDERE

Simone Chiocchetti (University of Cologne)
ADER schemes go faster and faster

Matej Klíma (University of Verona)
3D direct Arbitrary-Lagrangian-Eulerian ADER-DG schemes on moving meshes with connectivity changes

Daniel Regener Roig (University of Bergamo)
Robust discontinuous Galerkin solution of the multicomponent Euler model for multiphase flows

20:00 – ... **CONFERENCE DINNER**

Daily program: Thursday, 2 April 2026

09:15 **OPENING – ROOM MADRUZZO**

09:15 – 10:00 **Vincent Perrier** (University of Pau)

Discrete Hodge-Helmholtz decomposition and application to the accuracy problem in the computation of low Mach number compressible flows

10:00 – 10:30 **Vladimir Titarev** (Computing Center of RAS, Moscow)

Numerical study of some problems of rarefied gas mechanics

10:30 – 11:30 **COFFEE BREAK**

11:30 – 12:00 **Angelo Iollo** (University of Bordeaux)

Collocated Reduced-Order Models for High-Order Schemes

12:00 – 12:30 **Gabriella Puppo** (Sapienza University of Rome)

From kinetic equations to macroscopic models for two phase flow

12:30 – 13:00 **Andrea Thomann** (INRIA, Strasbourg)

Numerical methods for hyperbolic multi-scale systems

13:00 – 15:00 **LUNCH**

ROOM MADRUZZO

ROOM BELVEDERE

15:00 – 15:30 **Klaas Willems** (KU Leuven)

A semi-Lagrangian method for the polyatomic ESBGK model

Michal Žeravý (Slovak University of Technology, Bratislava)

A priori time limiting for the Quinpi scheme

15:30 – 16:00 **Silvia Preda** (University of Insubria, Como)

SCOUT: Semi-Lagrangian COnservative and Unconditionally Stable schemes for nonlinear advection-diffusion problems

Alessandra Zappa (University of Insubria, Como)

Extending Quinpi in 2D: an implicit scheme for nonlinear conservation laws

16:00 – 16:30 **Pierre Anguill** (CEA, Arpajon)

An asymptotic preserving method for the linear transport equation on general meshes

Roman Stuhlmacher (KTH Royal Institute of Technology)

Superconvergence through SIAC filtering on non-uniform, structured meshes

16:30 – 17:15 **COFFEE BREAK**

17:15 – 17:45 **Dean Yuan** (INRIA, Bordeaux)

Anisotropic mesh adaptation for the non-linear shallow water equations

Julio Careaga (University of Groningen)

Entropy preserving discontinuous Galerkin schemes for shallow water moment models

17:45 – 18:15 **Romane Helie** (CEA, Arpajon)

Lagrange-Remap scheme for 3D ideal magnetohydrodynamic equations in a Cartesian AMR hydrocode

Felipe Forte Tenreiro (INRIA, Bordeaux)

On the design of improved Boussinesq-type models for practical applications: a trade-off between accuracy and performance

17:45 – 18:15 **Galina Lazareva** (RUDN University, Moscow)

Mathematical model of current distribution in a tungsten plate during pulsed heating

Marina Yashina (MADI University, Moscow)

On the design of improved Boussinesq-type models for practical applications: a trade-off between accuracy and performance

Daily program: Friday, 3 April 2026

09:15 **OPENING – ROOM MADRUZZO**

09:15 – 10:00 **Eleuterio Francisco Toro** (University of Trento)

Time-reconstruction solvers for the GRPm. Application to ADER methods for hyperbolic systems.

10:00 – 10:30 **Walter Boscheri** (CNRS, Chambéry)

Structure preserving finite volume schemes on Voronoi grids: involutions, asymptotic limits, algebraic constraints and thermodynamics

10:30 – 11:30 **COFFEE BREAK**

11:30 – 12:00 **Evgeniy Romenski** (Sobolev Institute of Mathematics, Novosibirsk)

Numerical Evaluation of Effective Elastic Moduli of a Heterogeneous Deformed Fluid Saturated Porous Medium

12:00 – 12:30 **Francois Vilar** (Université de Montpellier)

Subcell monolithic ADER-AFE-DG/FV convex property preserving scheme

12:30 – 13:00 **Lucas Müller** (University of Trento)

A generalized Riemann problem-based compact reconstruction method for finite volume schemes

13:00 – 15:00 **LUNCH**

ROOM MADRUZZO

ROOM BELVEDERE

15:00 – 15:30 **Maurizio Tavelli** (University of Verona)

A structure-preserving semi-implicit four-split scheme for continuum mechanics

Julie Patela (CEA, Arpajon)

An APITALI approach for Lagrangian schemes enforcing the discrete entropy growth for Euler equations

15:30 – 16:00 **Ilya Peshkov** (University of Trento)

On various structures of continuum mechanics models and the need for structure preserving integrators

Marco Artiano (Johannes Gutenberg University Mainz)

On Affordable Entropy-Conservative and Entropy-Stable Methods for Nonconservative Hyperbolic Systems

16:00 – 16:30 **Michael Dumbser** (University of Trento)

A simple and general framework for the construction of exactly div-curl-grad compatible discontinuous Galerkin finite element schemes on unstructured simplex meshes

16:30 – 17:15 **CLOSURE AND COFFEE BREAK**

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Abstracts of oral presentations

Recent development on Active flux methodology

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ABSTRACT

Considering an hyperbolic problem defined on a domain Ω of \mathbb{R}^d , $d = 1, 2, 3$ covered by polygons, we are interested in methods that use globally continuous approximation of the data, while being able to handle discontinuities. One possible method use as degrees of freedoms the Gauss-Lobatto points on the boundary or the polygons K , and moments in K of the solution. From this, it is possible to use a globally continuous approximation where

- when the polygon is a triangle, for an approximation of order $k + 1$, one uses an approximation space which contains \mathbb{P}^k and is contained in \mathbb{P}^{k+1} , see [1]
- when a general polygon is used by using the approximation tools defined for Virtual Finite Elements, [2], see [3].

To illustrate, let us take the example of third order of accuracy, i.e. $k = 2$. On each face of the polygon, we have 3 Gauss-Lobatto point (defining a quadratic polynomial on this face), and we need one moment, the average value. The average is updated by

$$|K| \frac{d\bar{\mathbf{u}}_K}{dt} + \int_{\partial K} \mathbf{f}(u) \mathbf{n} d\gamma = 0$$

which is doable (without Riemann solver) because we know \mathbf{u} on the boundary, and the point values are updated by

$$\frac{d\mathbf{u}_\sigma}{dt} + \sum_{K, \sigma \in K} \Phi_\sigma^K(u) = 0.$$

The residuals are approximations of $\text{div } \mathbf{f}(\mathbf{u})$ on the polygon K which are consistent with this operator, in the sense that if the solution were linear, and the flux linear with constant Jacobian, the sum would exactly give $\text{div } \mathbf{f}(\mathbf{u})$. We have shown that this condition is essentially sufficient, together with the classical stability bounds, to get a Lax-Wendroff theorem. This kind of schemes is within the family of active flux, initially designed by Roe and his co-worker, see [5] for an early contribution, and has since then received a lot of attention: one may mention the work of Barsukow, Klingenberg, Heltzel in Europe, Wu, Kurganov and others in China. It turns out that this kind of scheme has a lot of connection with discontinuous Galerkin schemes. Using a Runge Kutta time discretisation, one can reinterpret the above scheme, for a linear flux with constant Jacobian as, for each Runge Kutta cycle, one step of DG followed by a projection of the discontinuous solution onto the solution space. The projector operator can be a simple arithmetic average (at least in 1D), or a more complex one with upwinding flavor, see [6]. We also have gone further by showing how to properly describe the implementation of boundary conditions in this kind of scheme in [7]. It is well known that solutions of hyperbolic problems are generically discontinuous. This implies that some non linear stabilization mechanism must be added. We have proposed a very simple algorithm to guaranty that the solution stays in the invariant domain in [3], and where in addition a non linear mechanism is proposed to remove spurious oscillations [7]. In the talk, we will make a coherent summary of all these findings.

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An asymptotic preserving method for the linear transport equation on general meshes – HONOM 2026

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Keywords: finite volume schemes; radiative transfer; asymptotic analysis; computational transport; monotone anisotropic diffusion; unstructured meshes.

ABSTRACT

The reference [1] presents a finite-volume scheme for a grey linear transport equation in 2D. This scheme satisfies the following properties:

consistency on general meshes; conservation of energy; asymptotic preserving accuracy from free-streaming to diffusion; and flexibility in the choice of the limit diffusion scheme.

Unfortunately, this scheme is not colocated. Thus, handling boundary conditions is difficult, and positivity of the solution is not guaranteed.

Therefore, we propose a new colocated method derived from this scheme to overcome these limitations. The scheme of [1] is positive in the diffusion limit, but not in transport regime. In contrast, an upwind scheme is positive in the transport regime but is not asymptotic preserving. Hence, in the present talk, we introduce the new method whose numerical fluxes are constructed as a hybridization of those associated with these two schemes.

The resulting scheme is implicit, and in order to avoid solving a global linear system of size equal to the number of cells times the number of directions, we employ a sub-cycling strategy. Moreover, the scheme involves a nonlinear diffusion discretization operator. Consequently, sub-cycling is also used to handle the nonlinearity of the diffusion scheme. Thanks to the properties of our diffusion scheme in 2D [2] and 3D [3], we enforce the solution to remain positive in the diffusion regime. With the hybrid fluxes, we also enforce the solution to remain positive in the free-streaming regime. We expect that an appropriate choice of the hybridization parameter will ensure positivity in all regimes.

We have conducted numerical tests on structured and unstructured 2D meshes, which assess that the expected properties are satisfied. In particular, we carried out the Lattice test problem to compare our results with existing methods, notably the Discontinuous Finite Element (DFE) and the method of [1].

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On Affordable Entropy-Conservative and Entropy-Stable Methods for Nonconservative Hyperbolic Systems

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Keywords: finite volume schemes; discontinuous Galerkin finite element schemes; structure-preserving methods, flux differencing; entropy conservative; entropy stable; nonconservative hyperbolic systems; summation-by-parts operators; curvilinear meshes; high order methods.

ABSTRACT

Modern entropy-conservative and entropy-stable (summarized as entropy-preserving) methods for hyperbolic conservation laws rely on Tadmor’s theory for two-point entropy-preserving numerical fluxes [1] and its higher-order extension via flux differencing using summation-by-parts (SBP) operators [2, 5]. Two-point formulations have been extended to nonconservative systems in [3] with follow-up generalizations to SBP methods [4, 6]. Here, we extend these works by proposing a reformulation of entropy-preserving two-point formulations for nonconservative hyperbolic systems. We analyze necessary and sufficient conditions, and obtain a full characterization of entropy-preserving three-point methods within the finite volume framework. This formulation is then extended in multiple space dimensions to SBP frameworks and to curvilinear meshes using discontinuous Galerkin (DG) methods. The proposed formulation enables an algorithmic construction of entropy-preserving methods, which we use to derive several new entropy-preserving schemes for nonconservative hyperbolic systems. Numerical experiments show the robustness and accuracy of the proposed schemes on various nonconservative hyperbolic systems, including the Euler equations in nonconservative form, magnetohydrodynamics (MHD) and shallow water equations.

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Design of High-Order Fractional-Step Runge–Kutta Methods with Low-Order Subintegrators

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Keywords: Time integration; Fractional Step Runge-Kutta; Additive Runge Kutta methods; High order methods; Operator Splitting; Strong stability.

ABSTRACT

Some of the most common Operator Splitting methods are the first-order Lie-Trotter (or Godunov) method and the second-order Strang (or Strang-Marchuk) method. Typically, the substeps are integrated using standard Runge-Kutta (RK) methods and variations of them. Recently, Spiteri & Wei introduced a formalism for writing the composition of RK methods within a fractional-step method as a larger RK method, herein referred to as an FSRK method and represented as an Additive RK method. This formalism allows the sophisticated tools developed for analysis of RK methods to be applied to FSRK methods as well, in order to understand and improve their stability and accuracy properties.

Typically, the order of the FSRK method is equal to the minimum of the local order of accuracy of all the substep integrators and the OS method. It is natural to ask whether, by a clever choice of substep integrators, some error cancellation might be designed into the method so that the order of the FSRK method is higher than the order of some of its component parts. We show that it is possible to construct FSRK methods whose overall order of accuracy is higher than the order of the subintegrators.

We present new families of such methods, and equip them with useful properties such as L-Stability, Strong Stability Preserving, and symplecticity. We also perform numerical tests to illustrate the behavior of the new methods constructed.

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A second order accurate generalized Riemann solver for a hyperbolic model of two-layer thin film flow

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Keywords: finite volume schemes; Generalized Riemann problem; high order methods; Riemann invariants; Thin film flows.

ABSTRACT

This work is concerned with developing a second-order generalized Riemann solver for a hyperbolic system which governs the first order dynamics of a two-phase thin film flow model under the influence of a perfectly soluble anti-surfactant solute. The hyperbolic system is a rich hyperbolic system as it possesses an entire class of entropies and can be decoupled along its Riemann invariants [1]. Extending the first-order Godunov approach, the solver is used to construct a temporal-spatial coupled second-order GRP-based finite-volume method [2]. Numerical experiments including comparisons to MUSCL finite-volume schemes with Runge-Kutta time stepping confirm the accuracy, efficiency and robustness of the higher-order ansatz.

The construction of GRP methods requires to compute temporal derivatives of intermediate states in the entropy solution of the generalized Riemann problem. These derivatives are obtained from the Rankine-Hugoniot conditions as well as a characteristic decomposition using Riemann invariants. Notably, the latter can be computed explicitly for the two-layer thin film model, which renders this system to be very suitable for the GRP approach. Moreover, it becomes possible to determine the derivatives in an explicit, computationally cheap way.

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Energy preservation properties of the Domain of Dependence stabilization method on cut cell meshes

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Keywords: discontinuous Galerkin schemes; cut cells; small cell problem; structure-preserving methods; high-order methods.

ABSTRACT

Cartesian cut cell meshes provide a simple procedure to generate accurate meshes for complex geometries by embedding object boundaries in a structured cartesian mesh and removing the intersection between the objects and any intersecting cartesian cells. While simple and accurate, the main drawback is the loss of control over size and shape of the resulting so called cut cells in the mesh, putting severe restrictions on the time step size when explicit Runge-Kutta schemes are applied. This is often called the small cell problem.

Among the many approaches that exist to handle the small cell problem, we will consider the Domain of Dependence stabilization method. It consists of penalty terms on top of a standard discontinuous Galerkin scheme enabling a time step size based on the cartesian structured mesh by establishing information flow directly between neighbors of small cells, lifting the time step size restrictions imposed by small cells.

We present an abstract framework for the DoD stabilization based on a splitting of numerical fluxes into central and dissipative parts which is energy preserving for linear equations. As a test example we consider the acoustic wave equation. We will also give a short discussion on a possible extension towards entropy preservation for nonlinear systems like the Euler equations.

Structure preserving finite volume schemes on Voronoi grids: involutions, asymptotic limits, algebraic constraints and thermodynamics

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Keywords: finite volume schemes; unstructured Voronoi grids; Structure-Preserving; Curl-free involution; Asymptotic limits; Thermodynamics compatibility.

ABSTRACT

We propose novel finite volume schemes on unstructured Voronoi meshes that are provably compatible with different structures of the underlying model of time-dependent nonlinear hyperbolic equations.

Firstly, a new curl-free and thermodynamically compatible finite volume scheme is devised to solve compressible heat conducting flows written in first-order hyperbolic form. The approach is based on the definition of compatible discrete curl-grad operators, exploiting the triangular nature of the dual mesh. We design a cell solver reminiscent of the nodal solvers used in Lagrangian schemes to discretize the evolution equation for the thermal impulse vector, and we demonstrate that the resulting numerical scheme ensures energy conservation, local nonnegative entropy production, as well as asymptotic consistency with the classical Fourier law in the stiff relaxation limit. A novel technique is proposed to transfer residuals from the dual to the primal mesh as subfluxes, which eventually yields the construction of entropy compatible semi-discrete methods.

Secondly, we tackle algebraic constraints by preserving the Geometric Conservation Law (GCL) for a unified model of continuum mechanics. We start by formulating the GCL at the continuous level, and subsequently we aim at satisfying this constraint by introducing the new concepts of general potential and generalized Gibbs relation. The new potential is nothing but the determinant of the distortion tensor, and the associated Gibbs relation is derived by introducing a set of dual or thermodynamic variables such that the GCL is retrieved by dot multiplying the original system with the new dual variables. Once compatibility of the GCL is ensured, thermodynamic compatibility is tackled in the same manner, thus achieving the satisfaction of a local cell entropy inequality. The two corrections are orthogonal, meaning that they can coexist simultaneously without interfering with each other. The compatibility of the new FV schemes holds true at the semi-discrete level.

The schemes and their properties are validated on a set of numerical test cases.

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Entropy preserving discontinuous Galerkin schemes for shallow water moment models

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Keywords: free-surface flow problem; energy equation; discontinuous Galerkin finite element schemes; entropy preserving schemes; high order methods.

ABSTRACT

In this talk, we present a derivation of an energy equation, and entropy variables for the so-called shallow water moment equations (SWME) [2, 3], which are an extension of the well-known shallow water model. In addition, we show that the standard Newtonian slip and Newtonian Manning friction terms are entropy dissipative with respect to the developed entropy variables. Based on the nodal discontinuous Galerkin spectral element method [1], we develop a high order entropy preserving scheme to approximate the SWME. For this, we construct a numerical flux following the derivation of the entropy variables in the continuous case, and such that an entropy preserving condition including non-conservative terms is fulfilled. Furthermore, we prove that the proposed scheme is also well-balanced. The numerical scheme is implemented in Julia making use of the library Trixi. Finally, numerical examples show the reliability of the scheme, validating the preservation of entropy and well-balance of the scheme as well as the expected order of convergence.

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HONOM 2026: Data-Driven Mimetic Framework for Turbulence Modeling

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Keywords: mimetic differences; high order; data-driven; turbulence modeling.

ABSTRACT

It is well recognized that the modeling of turbulent fluid flows poses several numerical challenges. The non-linear convective term of the Navier Stokes equation requires special care when performing the spatial discretization. Spatial operators that satisfy the discrete version of the continuous integration by parts property result in accurate discretizations for the convective term. Mimetic difference schemes [1, 2, 3] are based on obtaining discrete high order operators that are equivalent to the tensor calculus quantities of divergence and gradient. They possess a discrete equivalent of a global conservation law [4] (i.e., the extended Gauss divergence theorem), thereby leading to accurate discretizations of the nonlinear convective terms. In this work, we investigate the use of the high order mimetic operators in developing ground truth data as applied to the Burgers equation. This ground truth is then supplied as inputs to a neural network as velocity and gradients of velocity, and the network is trained to obtain a machine-learned subgrid scale (SGS) turbulence model. The performance of this machine-learned mimetic based model is compared with the prevalent SGS models such as the static & dynamic Smagorinsky models.

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ADER schemes go faster and faster

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Keywords: Discontinuous Galerkin schemes; ADER schemes; high performance computing; compute kernel design; low storage schemes.

ABSTRACT

The higher efficiency of ADER numerical schemes [1] with respect to similar semi-discrete methods is often cited as one of their main benefits, and attributed to their communication-avoiding design that makes them better suited for parallelism. Since 2006-2008 [2, 3], thanks to the introduction of a local DG predictor step replacing the Cauchy–Kovalewskaya procedure, these numerical methods offer greater versatility, making them well suited for applications to general PDE systems. This talk focuses on the algorithmic performance of the schemes on the node-level and simplifying their implementation.

In this work, we discuss how numerical methods of the ADER family can be pushed to the extreme for applications in fluid flow requiring high performance. We show how some appropriate choices of basis functions can be used to construct very simple and efficient space-time evolution operators that are valid on arbitrary geometries. We show how partial orthogonalization of the basis functions and sparsity patterns and in the space-time operators, initially studied in this work as a purely implementation-level optimization, can explicitly guide the construction of the scheme, giving rise to a family of methods from which the implementer can choose the “preferable” option depending on hardware-level considerations like minimizing storage requirements or minimizing floating point operations in the DG predictor step which constitutes the core of the scheme. Some basis functions are simpler to evaluate but lead to heavy, dense space-time evolution operators, while appropriately chosen orthogonal basis functions can lead to simpler evolution operators, but require additional storage, memory bandwidth and even compute for the simple evaluation. The talk tries to give some insights on the co-design process of these numerical methods where implementation takes a central role.

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On the heat GLM system and its compatible discretization

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Keywords: augmented heat-GLM system; symmetric hyperbolic systems; Hamiltonian systems; variational principles; asymptotic-preserving (AP) schemes

ABSTRACT

The Generalized-Lagrangian Multiplier (GLM) cleaning, introduced by Munz *et al.* in [1] and further studied, e.g., in [2, 3, 4], is a practical strategy for enforcing div- and curl-free constraints. In this talk, I will present a rigorous formulation of the GLM curl cleaning for a prototype heat propagation model, analyzing how the introduction of cleaning variables modifies the system’s mathematical structure and asymptotic behavior. I will show that the equations can be rigorously derived from an underlying variational principle and that the resulting system is symmetric hyperbolic. Under suitable boundary conditions, the system preserves the initial curl of its vector fields at the continuous level and is asymptotically consistent with Fourier’s law of heat conduction. These structural properties serve as a blueprint for the development of a compatible semi-implicit discretization on staggered grids, which provably preserves the asymptotic Fourier limit, the Lyapunov-function decay, and the curl-free involution of the heat flux variable. The theoretical analysis is supported by a set of carefully designed test cases confirming the scheme’s properties.

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On hyperbolic approximations for a class of dispersive and diffusive-dispersive equations

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Keywords: Relaxation approaches, dispersive and diffusive-dispersive equations, generalized KdV equations, non-classical shock waves.

ABSTRACT

We introduce new approximate systems for dispersive and diffusive-dispersive equations with nonlinear fluxes. For purely dispersive equations, we construct a first-order, strictly hyperbolic approximation. Local well-posedness of smooth solutions is achieved by constructing a unique symmetrizer that applies to arbitrary smooth fluxes. Under stronger conditions on the fluxes, we provide a strictly convex entropy for the hyperbolic system that corresponds to the energy of the underlying dispersive equation. To approximate diffusive-dispersive equations, we rely on a viscoelastic damped system that is compatible with the found entropy for the hyperbolic approximation of the dispersive evolution. For the resulting hyperbolic-parabolic approximation, we provide a global well-posedness result. The structure of the new approximate systems allows to apply standard numerical simulation methods from the field of hyperbolic balance laws. We confirm the convergence of our approximations even beyond the validity range of our theoretical findings on a set of test cases covering different target equations. We show the applicability of the approach for strong nonlinear effects leading to oscillating or shock-layer-forming behavior.

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Volume Term Adaptivity for Discontinuous Galerkin Schemes

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Keywords: Discontinuous Galerkin; Adaptive Methods; High-Order; Entropy Stability

ABSTRACT

We introduce the concept of volume term adaptivity for high-order discontinuous Galerkin (DG) schemes solving time-dependent partial differential equations. Termed v -adaptivity, we present a novel approach that exchanges the discretization of the the volume contribution of the DG scheme at every Runge-Kutta stage based on the local element entropy production. The main goal is to use the cheap weak form (WF) volume integral whenever possible to reduce computational cost.

Depending on the choice of the indicator, the v -adaptive DG scheme improves robustness, efficiency and approximation quality compared to baseline schemes with a uniform volume term discretization. The present methodology can be perceived as a generalization of [1] wherein the flux-differencing (FD) volume term is blended with a finite volume (FV) scheme on the DG subcells to stabilize the simulation. In this work, we consider pairing the WF volume term with the FD volume integral, both standard and stabilized version.

We present results for the compressible Euler and Navier-Stokes equations in two and three dimensions using a high-order discontinuous Galerkin Spectral Element Method. These results demonstrate that based on the tolerated local entropy production, robustness and/or efficiency of the scheme can be improved for challenging simulations.

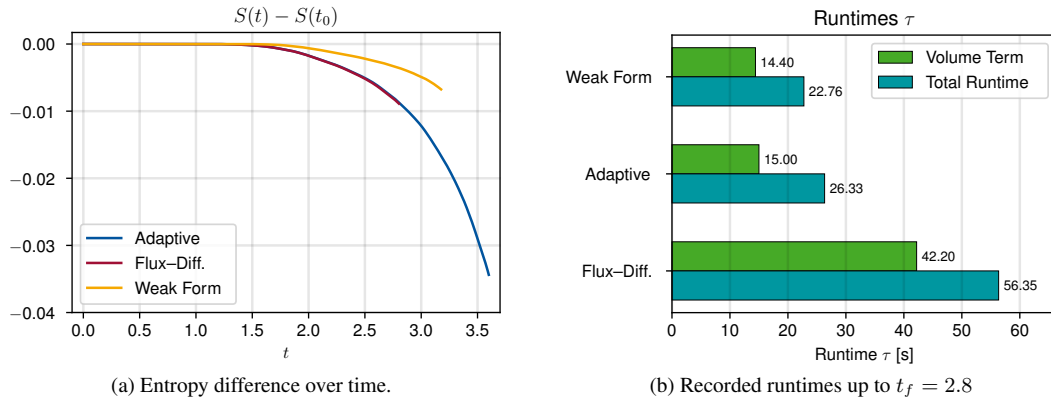


Figure 1: Kelvin-Helmholtz instability testcase on a 32×64 triangular mesh with $p = 3$.

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A polytopal exterior calculus framework with application to the Maxwell equations on manifolds

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Keywords: de Rham complex; polytopal method; exterior calculus; Maxwell equations; PDE on manifolds.

ABSTRACT

The properties of the de Rham complex (here, on an open set $\Omega \subset \mathbf{R}^3$)

$$\{0\} \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}, \quad (1)$$

are essential to proving the well-posedness, and constraint preservation, of several models of partial differential equations – in particular those encountered in the modelling of electromagnetic fields. Designing robust schemes that preserve the same structures as the continuous model is therefore best done through discrete versions of this complex. A direct discretisation of (1) however often involves separate treatments of each space and differential operator. Instead, the de Rham complex can be recast in an exterior calculus framework, as a complex of differential forms linked by the exterior derivative:

$$\dots \xrightarrow{d^{k-1}} H\Lambda^k(\Omega) \xrightarrow{d^k} H\Lambda^{k+1}(\Omega) \xrightarrow{d^{k+1}} \dots \quad (2)$$

This presentation allows for a dimension-independent treatment of the complex (2D and 3D alike), and for a unified construction for all spaces and differential operators – which have a similar nature and description for all form degree k .

The efficiency of this approach through differential forms has been illustrated, for finite elements, by the Finite Element Exterior Calculus [1]. Because of their search for globally conforming shape functions, finite element methods however face mesh restrictions that can hinder their efficiency (e.g., in terms of required number of degrees of freedom, or local mesh refinement).

We will present a discrete version of the de Rham complex of differential forms, the Discrete De Rham method (DDR), that is of arbitrary order of accuracy and applicable on meshes made of generic polygons/polyhedra [2]. We will show that adopting the exterior calculus framework facilitates the adaptation of DDR to the case where Ω is a manifold, and we will illustrate its use to design a constraint-preserving “polygonal” scheme for Maxwell’s equations on a surface [3].

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Uncertainty quantification of conservation laws with several random parameters

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Keywords: finite volume schemes; uncertainty quantification; low-rank tensors

ABSTRACT

This work addresses uncertainty quantification (UQ) for conservation laws with random initial conditions. We use the stochastic finite volume method (SFV), where the UQ problem is reformulated as a deterministic problem with additional variables [1]. To keep the computational cost reasonable, in [2] the SFV was combined with low-rank tensors. We introduce in [3] a new format combining full tensors for the physical space and tensor trains for the stochastic space. This hybrid format can easily be applied to any type of finite volume schemes. In this talk, I will present numerical experiments with one dimension in physical space and several dimensions in stochastic space. The Burgers' equation and the Sod problem with random initial conditions are solved using a MUSCL scheme. We compare the format proposed in [2] and the new hybrid format. We then solve the Shu-Osher problem, and show that the hybrid format is able to capture complex features of hyperbolic problems such as shocks and waves interactions.

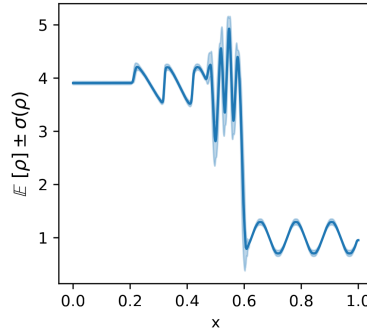


Figure 1: Mean density for the Shu-Osher problem with random initial conditions.

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A simple and general framework for the construction of exactly div-curl-grad compatible discontinuous Galerkin finite element schemes on unstructured simplex meshes

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Keywords: structure-preserving schemes; continuous-discontinuous Galerkin finite element schemes; high order methods.

ABSTRACT

We introduce a new family of discontinuous Galerkin (DG) finite element schemes for the discretization of first order systems of hyperbolic partial differential equations (PDE) on unstructured simplex meshes in two and three space dimensions that respect the two basic vector calculus identities exactly also at the discrete level, namely that the curl of the gradient is zero and that the divergence of the curl is zero. The key ingredient here is the construction of two compatible discrete nabla operators, a primary one and a dual one, both defined on general unstructured simplex meshes in multiple space dimensions. Our new schemes extend existing cell-centered finite volume methods based on corner fluxes to arbitrary high order of accuracy in space. An important feature of our new method is the fact that only two different discrete function spaces are needed to represent the numerical solution, and the choice of the appropriate function space for each variable is related to the origin and nature of the underlying PDE. The first class of variables is discretized at the aid of a discontinuous Galerkin approach, where the numerical solution is represented via piecewise polynomials of degree N and which are allowed to jump across element interfaces. This set of variables is related to those PDE which are mere consequences of the definitions, derived from some abstract scalar and vector potentials, and for which involutions like the divergence-free or the curl-free property must hold if satisfied by the initial data. The second class of variables is discretized via classical continuous Lagrange finite elements of approximation degree $M = N+1$ and is related to those PDE which can be derived as the Euler-Lagrange equations of an underlying variational principle. The primary nabla operator takes as input the data from the FEM space and returns data in the DG space, while the dual nabla operator takes as input the data from the DG space and produces output in the FEM space. The two discrete nabla operators satisfy a discrete Schwarz theorem on the symmetry of discrete second derivatives. From there, both discrete vector calculus identities follow automatically. We apply our new family of schemes to three hyperbolic systems with involutions: the system of linear acoustics, in which the velocity field must remain curl-free and the vacuum Maxwell equations, in which the divergence of the magnetic field and of the electric field must remain zero. In our approach, only the magnetic field will remain exactly divergence free. As a third model we study the Maxwell-GLM system of Munz *et al.*, which contains a unique mixture of curl-curl and div-grad operators and in which the magnetic field may be either curl-free or divergence-free, depending on the choice of the initial data. In all cases we prove that the proposed schemes are exactly total energy conservative and thus nonlinearly stable in the L^2 norm.

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HONOM 2026 High Order Mimetic Differences Curl Vector Calculus Identities

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Keywords: mimetic differences; curl operator; interpolation; divergence operator; Maxwell's equation; time discretization; high order methods.

ABSTRACT

We construct mimetic differences [1, 2, 3] high-order discrete analogs of 2D and 3D curl operators. These operators are built on a Cartesian staggered grid, utilizing the coordinate-free definition of the curl operator. It is shown that this definition allows a formula that relates each component of the curl to be computed utilizing a 2D divergence discrete analog applied to some shifts of the mesh points. Our approach is different from the ones given in [7, 1, 8]. These constructions do not use interpolations and they are defined for an arbitrary number of cells in any direction provided a minimum number of cells is provided in order to achieve the degree of accuracy required. It is shown that these operators satisfy vector calculus curl identities. In addition, Maxwell's equations can be solved accurately utilizing these curl operators together with the Mimetic Operators Library Enhanced (MOLE) [4]. We also aim to preserve mass, momenta and energy for Maxwell's equations [5].

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A discrete equivalence in HTC schemes for multiphase flows

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Keywords: Compressible multiphase flows; finite volume schemes; continuum mechanics; structure-preserving.

ABSTRACT

In the context of multiphase flows, the choice of governing equations, whether expressed in dimensional or non-dimensional state variables, often leads to discrepancies at the discrete level, despite their equivalence at the continuous level. This work addresses the development of a monotone and positivity-preserving discretization within the framework of Hyperbolic Thermodynamically Compatible (HTC) schemes [1].

We consider the evolution of the non dimensional volume fraction and show that stability and monotonicity require a specific CFL condition together with appropriately designed numerical viscosity coefficients. Then, the main contribution of this study is the derivation of a discrete product rule that guarantees mathematical equivalence between the evolution equations written in dimensional and non-dimensional variables. By enforcing this discrete consistency, the proposed scheme preserves the physical bounds of the state variables while maintaining the mathematical structure and equivalence of the model. The talk aims to give some perspectives on extending HTC schemes to satisfy additional structural properties.

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On the design of improved Boussinesq-type models for practical applications: a trade-off between accuracy and performance

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Keywords: Boussinesq-type models; Fully nonlinear weakly dispersive models; Discontinuous Galerkin methods; Higher-order methods; High-performance computing.

ABSTRACT

With the ever-increasing power of computing resources, developing codes capable of accurately handling *large-scale* applications is of the utmost interest to the research community. While the complex hydrodynamic description up to the shoreline can be greatly simplified in the context of overflow flooding by adopting the Saint-Venant equation model, this is unfortunately not the case in overtopping scenarios. Boussinesq-type (BT) models are essentially depth-averaged shallow-water approximations to the fully dispersive and nonlinear water wave problem with additional dispersive terms.

The strategy employed here consists of decoupling the hydrostatic and dispersive parts of the equations, thereby generating a source term that requires solving *at least* one elliptic problem. However, since the inversion of a *time-dependent* second-order differential operator can become computationally prohibitive, we opted to assess how the Serre-Green-Naghdi (SGN) models proposed by Lannes *et al.* [1], for which this time-dependency *constraint* has been relaxed, would fare against highly-dispersive nonlinear waves. For this, we employ a high-order Discontinuous Galerkin (DG) scheme.

In the figure presented below (1), we show how this set of SGN equations can be used to simulate the propagation of highly-dispersive periodic waves past a submerged bar (cf. [2]). In addition, we have also seen that they can be made to handle nonlinear shoaling waves by tuning the regularisation terms responsible for the instabilities observed during wave breaking or upon reflection on a wall (not presented here for brevity).

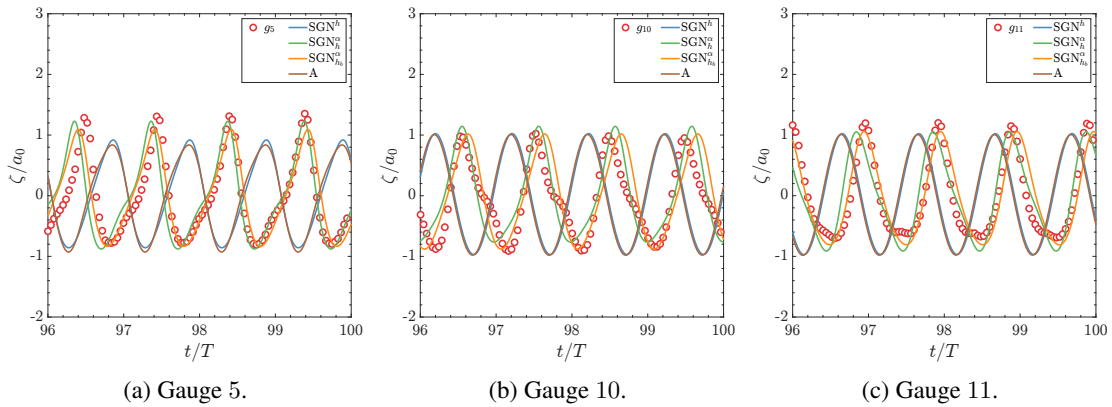


Figure 1: Comparison between experiments (○) and computations (—) for the normalised free-surface elevation on the first (1a) and last two wave gauges (1b and 1c) on the back of the bar ($\mu|k| \sim 1.69$).

Implicit methods for numerical solution of hyperbolic systems given by transport equations with nonlinear sorption

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Keywords: hyperbolic transport equations, sorption, implicit time discretizations.

ABSTRACT

We are interested in the systems of PDEs written in the form,

$$\partial_t \Theta(\mathbf{c}) + \partial_x \mathbf{c} = 0, \quad (1)$$

where $\mathbf{c} = \mathbf{c}(\mathbf{x}, t) \in \mathcal{R}^m$ are unknowns (the concentrations) and $x \in \mathcal{R}$, $t \geq 0$. Typical applications are contaminant transport [2] and liquid chromatography [1], where the substances are transported in porous media, and the given function Θ describes the equilibrated interaction of the substances with the stationary phase (the adsorption).

Under some assumptions on the vector function Θ and its Jacobian matrix Θ' , one can treat (1) as a hyperbolic system [1]. If the transformation $\mathbf{q} = \Theta(\mathbf{c})$ is applied and the inverse function $\mathbf{F} := \Theta^{-1}$ is available, then one obtains from (1) the equivalent form,

$$\partial_t \mathbf{q} + \partial_x \mathbf{F}(\mathbf{q}) = 0, \quad (2)$$

and some standard numerical methods for hyperbolic systems can be used.

Very often, one has to treat the problem (1) directly when the values of \mathbf{c} must be found numerically for each argument \mathbf{q} using an iterative nonlinear algebraic solver for the equations $\Theta(\mathbf{c}) = \mathbf{q}$. In that case, even if explicit time discretization is used, one has to solve for each discrete value $\mathbf{c}_i^n \approx \mathbf{c}(\mathbf{x}_i, t^n)$ at discrete points (\mathbf{x}_i, t^n) a system of $m \times m$ algebraic equations. Consequently, numerical methods based on implicit time discretization, with some compact stencils that enable fast algebraic solvers like the fast sweeping methods [6], are an attractive alternative to explicit methods that are only conditionally stable. We present unconditionally stable implicit methods for certain specific problems of the form (1).

First, if the form (2) is available, we use the compact high-resolution implicit scheme as given in [3] for some 1D hyperbolic systems, in [8] for 2D shallow water systems with topography, and in [5] for some scalar equations in 2D. The advantage of the method is its unconditional stability and second-order accuracy for smooth solutions that can be reached through a finite number of iterations. Moreover, a high-resolution form is available that is provably non-oscillatory for linear scalar advection with divergence free velocity [5]. The property is valid for any time step; however, for “fast waves”, if present in the solution of (2), additional limiting in time must be applied.

Second, if one has to work directly with (1), because the inverse function \mathbf{F} is not available, we present an interesting option to use numerical methods based on the inverse Lax-Wendroff procedure for 1D advection on networks [4]. The advantage of this approach is that the limiting in time need not be applied for large time steps, but only for very small ones when the “slowness” is too large. Moreover, a nonuniform spatial mesh can be used easily, simplifying the treatment of coupling conditions for the vertices of the network.

Finally, we present a direct application of the compact implicit high-resolution method for (1) which can straightforwardly treat the nonlinear terms after the time derivative [7]. Moreover, some preliminary results for hyperbolic balance laws using a well-balanced implicit method based on equilibrium variables [9] might be presented.

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High order accurate ADER Finite Volume schemes based on novel genuinely multidimensional Riemann solvers

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Keywords: Finite Volume (FV) schemes, Residual Distribution (RD) methods, hyperbolic equations, genuinely multidimensional Riemann solvers, multidimensional Osher scheme, N scheme, unstructured polygonal meshes.

ABSTRACT

In this talk we present two simple but complete genuinely multidimensional Riemann solvers for non-linear systems of hyperbolic conservation laws on general unstructured polygonal Voronoi-like tessellations [1].

The first method is a direct extension of the Osher-Solomon Riemann solver to multiple space dimensions. Here, the multi-dimensional numerical dissipation is obtained by integrating the absolute value of the flux Jacobians over a dual triangular mesh around each node of the primal Voronoi mesh. The second method is a multidimensional upwind flux based on the so-called N-scheme, originally developed in the framework of residual distribution. Both methods use the full eigenstructure of the underlying hyperbolic system and are therefore complete by construction.

Then, a simple higher order extension up to fourth order in space and time is introduced at the aid of a CWENO reconstruction in space and an ADER approach in time, leading to a family of high order accurate fully-discrete one-step schemes based on genuinely multidimensional Riemann solvers.

We close the talk by showing applications of our new numerical schemes to several test problems in the field of the compressible Euler equations. In particular, we show that the proposed schemes are at the same time carbuncle-free and preserve certain stationary shear waves exactly.

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Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency.

Lagrange-Remap scheme for 3D ideal magnetohydrodynamic equations in a Cartesian AMR hydrocode

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Keywords: Ideal MHD equations; Solenoidal method; Finite-Volume schemes; Godunov-type method; Lagrange-remap scheme; High-order accuracy.

ABSTRACT

We present the high-order numerical scheme implemented in the cartesian AMR hydrocode HERA developed at CEA for ideal 3D MHD equations. The discretized model [1] contains two magnetic field instances: a centered one, denoted \mathbf{B} , which contributes to the magnetic energy, and a staggered one, denoted \mathbf{C} , which preserves the divergence. Its resolution is splitted into two steps:

1. We first solve the modified ideal MHD system in which some instances of the magnetic field are frozen to \mathbf{C} on faces using a dimensional splitting and a Lagrange-remap approach. At first-order, the 1D Lagrangian scheme is a generalization to MHD of the two states Godunov acoustic solver [2] and is entropic under CFL condition. The high-order extension is a generalization of the GAIA scheme proposed in [3] for hydrodynamics. The 1D remap step is based on polynomial reconstructions [4].
2. We then update the face-valued \mathbf{C} of the magnetic field using a constrained transport type method. During this step $\nabla \cdot \mathbf{C}$ is preserved at machine precision.

We will also discuss the coarsening and refinement AMR steps and conclude by presenting numerical results on standard test-cases in 2 and 3 dimensions.

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Novelties on arbitrary high-order two-dimensional composite finite volume schemes

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Keywords: finite volume schemes; Eulerian methods; high order methods.

ABSTRACT

We deal with extension of the finite volume classical ARS (Approximate Riemann Solver) of conservation laws in dimension $d = 2$:

$$\partial_t \mathcal{U} + \operatorname{div}_{\mathbf{x}} \mathcal{F}(\mathcal{U}) = 0, \quad \mathbf{x} \in \mathbb{R}^2.$$

We consider composite finite volume schemes in the sense that the numerical fluxes are well defined (consistent and locally conservative) at all co-dimension mesh object (nodes and edges here).

First part is devoted to geometrical properties of normals at such locations. For each of these mesh items, we propose an extension in the construction of local normals [1] [2] in such a way to fulfill (some necessary) LSV conditions: Local conservation, Stokes formula and cell Volume formula. The new normal corner vector \hat{C}_j^r are based on standard C_j^r (e.g. [1]) and some other complementaries vectors D_j^r . They possess properties that do not break LSV conditions and also show some rotational properties (orthogonality with C_j^r in each cell and around nodes). Up to our knowledge, this is a new result for finite volume methods on unstructured meshes making a link with discrete Helmholtz-Hodge decomposition. Moreover, they still permit to obtain arbitrary order quadrature (composite) formula.

The second part is devoted to extending classical edge schemes (Roe, VFFC, Rusanov, HLL ..) at node in such a way that consistency and local conservation is obtained. The resulting composite node/edge scheme offer another way of obtaining a multi-dimensional aware scheme (see also [3]). For Euler equations of compressible gas dynamics, we use arbitrary order composite formula flux and a valid high order spatial reconstruction obtained with an induced limitation process [4]. We will show numerical results to assess the accuracy and robustness of some composite schemes.

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Collocated Reduced-Order Models for High-Order Schemes

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Keywords: low-order models, high-order methods.

ABSTRACT

We present a mathematical framework for collocation-based reduced order models applicable to high-fidelity, high-order schemes. We introduce a sparse collocation strategy that selects a minimal set of grid indices to ensure that the quadrature of inner products over training snapshots remains consistent with the underlying reduced basis space. The core of this approach is a non-negative least squares (NNLS) quadrature rule and an associated sparse collocation projector Π_c . This projector is realized via a restriction operator P_c and weights W , satisfying the approximate snapshot consistency $\langle \phi_m, u_k \rangle_W \approx \langle \phi_m, u_k \rangle$ for trial modes ϕ_m and snapshots u_k , resulting in an oblique interpolatory reconstruction.

Our analysis formalizes the algebraic properties of the resulting collocation operator. We demonstrate, for example, that for linear problems with skew-self-adjoint evolution operators, the reduced dynamics can be cast in a generalized Hamiltonian form that preserves discrete energy. Specifically, the discrete Hamiltonian

$$H(u, v) = \frac{1}{2} \|v\|_G^2 - \frac{1}{2} \langle u, P_c^T L P_c u \rangle$$

is conserved along trajectories of the reduced system, provided the reduced stiffness operator is symmetric in the G -inner product. Here, $G = P_c^T P_c$ arises as the Gram matrix of collocated basis evaluations and L the self-adjoint space discretization operator.

We derive general stability bounds in energy norms associated with the projection and NNLS perturbation errors, leading to conditional stability akin to classical CFL constraints. Under vanishing residual error $\varepsilon_{\text{NNLS}}$, convergence to the full projection is recovered in the limit of dense collocation. The framework is validated using an exact in time fourth-order in space wave equation scheme and a Lax-Wendroff scheme for the shallow-water equations.

This framework is based on [1] and [2].

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An Entropy Stable Discontinuous Galerkin Spectral Element Method on Heterogeneous Grids

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Keywords: discontinuous Galerkin schemes; heterogeneous grids; entropy stable; high order methods.

ABSTRACT

The discontinuous Galerkin spectral element method (DGSEM) has proven to be a highly efficient numerical scheme for the approximation of smooth multiscale problems, e.g., turbulence in the present case. Key aspects for the high efficiency are its excellent parallelization properties and the use of a tensor-product ansatz, where the multidimensional space operator is decomposed into a sequence of one-dimensional operations. Efficient and robust application-oriented large eddy simulations (LES) of turbulence were enabled by the use of entropy-stable DGSEM in a flux differencing formulation. However, for the discretization of complex geometries, more flexible element types, e.g., simplexes, are in general more favorable. In a recent work, Montoya and Zingg [1] presented an efficient entropy stable flux differencing DGSEM formulation on triangular and tetrahedral elements. To benefit from the computational efficiency of the tensor-product ansatz, the authors used a collapsed coordinate transformation on Legendre–Gauss–Lobatto and Legendre–Gauss–Radau nodes. The authors were able to show that their scheme is computationally more efficient as previously developed entropy stable schemes which utilize a full-order basis. Moreover, the intrinsic drawback of small time steps in the collapsed coordinates formulation was resolved by the temporal evolution of modal degrees of freedom (DOFs) instead of nodal ones.

In practical applications such as the flow around an airfoil in the majority of the computational domain, the use of, e.g., hexahedral elements can be favorable and only in a small subset, e.g., simplexes are required. For this, the approach in [1] is extended to triangular prism, pyramid and tetrahedral elements on Legendre–Gauss nodes to enable an efficient entropy stable DGSEM on heterogeneous grids. To avoid any load imbalances due to the additional modal switching of the residual on non-hexahedral elements, a time measurements based partitioning technique is employed for the load balancing.

In the present talk, an entropy stable DGSE scheme on heterogeneous meshes is presented [2]. A particular focus is placed on the collapsed coordinates transformation, the interelement flux computation between different elements types, the modal time-update of the non-hexahedral elements, and the load balancing strategy. The free-stream preservation and the convergence properties are demonstrated based on numerical experiments and the excellent scaling properties of the DGSEM are shown. This talk concludes with several application-oriented numerical problems.

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3D direct Arbitrary-Lagrangian-Eulerian ADER-DG schemes on moving meshes with connectivity changes

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Keywords: discontinuous Galerkin finite element schemes; ALE methods; high order methods; mesh optimization.

ABSTRACT

Enabling changes in the mesh connectivity between time steps significantly improves the robustness of Arbitrary Lagrangian-Eulerian (ALE) schemes in high shear scenarios. Mesh tangling can be effectively prevented with this strategy combined with mesh smoothing. In our work, we implement the high order ADER discontinuous Galerkin method on moving Voronoi-like polyhedral elements constructed from tetrahedral generator grids. The scheme integrates the governing hyperbolic PDE system (e.g. compressible Euler equations) using space-time control volumes [1]. These volumes are described by trilinear mapping of the mesh from old to new time level in regions where connectivity is preserved. We then construct *hole-like* elements which describe the connectivity transformations [2]. These are generally 4D space-time elements that have *non-zero* spatial volume *only* in the intermediate states between the old and new time steps.

All transformations of the tetrahedral generator grid connectivity that preserve the number of generator points can be decomposed into sequences of elementary 2-3 or 3-2 flip operations. Therefore, we focus on describing the control volumes that correspond to these elementary transformations on the generator grid. This includes both the *hole-like* elements and their neighbors which have changing numbers of face edges. We also present a mesh optimization algorithm designed to maintain mesh quality during vortical movements and to prevent the formation of sliver tetrahedra with high aspect ratios, which often precede the tangling of the generator mesh. Since we are limited to one flip per time step locally, this approach presents *unique challenges*: the mesh is transformed in small steps while the optimization target is constantly moving. The efficiency of the method is tested using a vortex-like movement within a cubic domain. Our results demonstrate that the *geometric conservation law is respected* and that total volume is conserved for both constant cases and changing connectivity cases.

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A structure-preserving compact high-order method for multi-dimensional hyperbolic conservation laws

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Keywords: compressible Euler equations, Active flux, high order methods.

ABSTRACT

Multi-dimensional conservation laws possess many more phenomena than their one-dimensional counterparts, for example turbulent flow with vortices and non-trivial stationary states. At the same time, computational resources are limited, and refinement is particularly costly for multi-dimensional simulations.

In this talk we propose a new hybrid finite element - finite volume method that achieves upwinding by locally evolving continuous data, instead of solving a Riemann problem. Its degrees of freedom are cell averages/moments and point values located at cell interfaces, the latter being shared between adjacent cells. This method can be found in the literature under the name of *Active Flux* and has been shown to achieve superior results on coarse grids, because it is structure preserving.

We shall present this numerical method applied to multi-dimensional systems of conservation laws, in particular the multi-dimensional Euler [1] and MHD-equations [2] and we shall also analyze the structure preservation properties of Active Flux [3].

This is joint work among others with *Wasilij Barsukow* (CNRS/University of Bordeaux, France) and *Junming Duan* (Chinese University of Hongkong).

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A p-adaptive discontinuous Galerkin spectral element method on heterogeneous, curvilinear meshes

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Keywords: high order methods; discontinuous Galerkin scheme; heterogeneous curvilinear grids; entropy stable; p-adaptivity.

ABSTRACT

The discontinuous Galerkin spectral element method (DGSEM) has proven to be a highly adaptable numerical scheme for the approximation of partial differential equations. The utilization of polynomials following a tensor-product ansatz within each element together with exclusive coupling of adjacent elements via their numerical flux results in high computational efficiency and excellent scaling properties. Although the DGSEM was initially developed for quadrilateral and hexahedral elements, this restriction was recently resolved via a collapsed coordinate transformation in combination with time evolution of the modal degrees of freedom instead of the nodal degrees of freedom in order to alleviate the arising time step restriction [1, 2].

However, the polynomial representation of the solution suffers from stability issues when a low-order approximation is used to simulate physical multiscale phenomena such as underresolved turbulence. At the same time, utilizing high-order polynomials throughout the entire domain comes with a significant time-step restriction and is often computationally prohibitive. By locally adjusting the polynomial degree of the approximation, a p-adaptive DG scheme can choose the appropriate amount of fidelity in regions of smooth quasi-laminar flow and sections with high turbulent intensity. However, implementations of p-adaptive DGSE schemes were so far limited to quadrilateral and hexahedral elements.

To overcome this restriction, we present the application of an entropy-stable p-adaptive DGSE operator to heterogeneous, curvilinear grids composed of hexahedral, prismatic, tetrahedral and pyramidal elements. Special care was taken to ensure freestream preservation and high-order convergence properties through careful handling of the numerical flux computation on curved grids. We present numerical results illustrating the accuracy and computational efficiency of this approach.

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EBR Schemes: Recent Developments

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Keywords: CFD, unstructured mesh, quasi-1D method, edge-based reconstruction.

ABSTRACT

The paper presents a brief overview of the developments in higher-accuracy EBR (Edge-Based Reconstruction) schemes completed over the past 5-6 years. The EBR schemes introduce a vertex-centered finite volume method that uses quasi-one-dimensional reconstructions of variables on edge-oriented extended stencils (Fig. 1) to improve accuracy in solving conservation-law problems on unstructured mixed-element meshes. Higher accuracy is achieved due to implementing the property of the method to reduce to high-order algorithm on translationally invariant meshes. For shock capturing, the approach offers EBR-TVD, EBR-WENO [3] and EBR-MP [2] schemes. The latter is one of the most recent achievements. The second notable achievement is the introduction of curvilinear reconstructions [2] used in prismatic boundary layers of the mesh. The use of such reconstructions is especially important when solving external flow problems. To solve real complex problems, it may be advisable to combine different types of reconstruction.

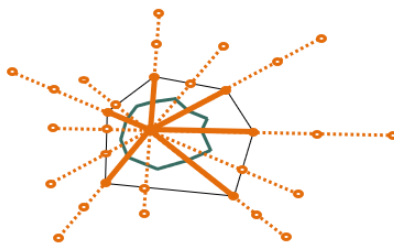


Figure 1: Extended edge-oriented stencils.

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Implicit high-order shock tracking for the compressible Euler equations using an extended discontinuous Galerkin method

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Keywords: (extended) discontinuous Galerkin, implicit shock tracking, level set, supersonic flow.

ABSTRACT

We present a novel high-order shock tracking method, which is capable of sharp representation of shock fronts in supersonic flows. The shock position is implicitly represented by a level-set. A local enrichment, also known as the extended Galerkin (XDG) approximation, enables a sharp representation of the discontinuous solution.

At the core of our method [1] lies a constrained optimization problem, where both, the degrees-of-freedom of the flow approximation and the interface position, serve as optimization variables which minimize spurious oscillations. This iterative computation of shock-aligned solutions employs a Sequential Quadratic Programming (SQP) method and is augmented with several measures to ensure robustness.

We are going to discuss the details of the method and present several numerical examples, e.g., the formation of steady bow shocks, cf. Figure 1, and shock-acoustic interactions.

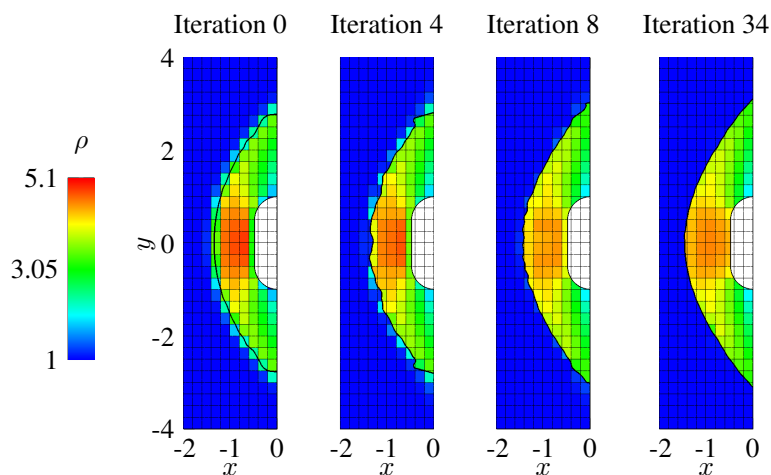


Figure 1: Compressible Euler for the Mach 4 bow shock. Selected iterations for the density ρ of a 10×32 mesh are shown. Starting from an initial guess, the shock interface (*left thick black line*), represented by a cubic spline level set, and the Euler solution are optimized.

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Local Characteristic Decomposition-Based Central-Upwind Flux Splitting

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Keywords: finite difference schemes; flux splitting; central upwind method; high order methods.

ABSTRACT

High-order finite-difference weighted essentially non-oscillatory (WENO) schemes have gained popularity thanks to their robustness and ability to achieve high resolution of complicated structures arising in the solution of hyperbolic systems of conservation laws. Stability of the finite-difference WENO schemes is typically enforced using a flux splitting, which can be conducted in several possible ways. The simplest flux splitting is the Lax-Friedrichs one, which is very robust, but suffers from excessive numerical dissipation. I will introduce a new robust central-upwind (CU) flux splitting, which leads to much sharper results without sacrificing the stability. The new splitting becomes even more accurate when performed in the local characteristic space. This allows each flux component of the system to use its own local characteristic speed, which significantly enhances the achieved resolution. The performance of the proposed scheme is demonstrated by simulating the one- and two-dimensional Euler equations of gas dynamics.

Advances in the Construction of Summation-By-Parts Operators on Arbitrary Grids and General Function Spaces

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Keywords: summation-by-parts operators; general function spaces; finite difference methods; stability; high order methods.

ABSTRACT

The accurate and robust numerical solution of hyperbolic conservation laws remains a central challenge in computational science. Entropy stability is crucial for ensuring the robustness of such numerical methods. Summation-by-parts (SBP) operators provide a general framework to systematically develop entropy-stable and high order schemes by mimicking continuous properties on a discrete level. They have proven to be a powerful tool for providing stable and high-order accurate numerical solutions. Classically, SBP operators are developed to differentiate polynomials up to a certain degree exactly. However, in many cases alternative function spaces such as trigonometric, exponential, or radial basis functions (RBFs) better approximate the underlying solution space.

The recently developed concept of function space SBP (FSBP) operators [1] provides a framework for SBP operators based on general function spaces, which has been extended to multiple dimensions in [2]. For instance, they enable provably conservative and stable RBF methods for numerically solving conservation laws. An optimization-based construction procedure for FSBP operators in one spatial dimension has been developed in [3].

First, I discuss extensions of this optimization problem, which are of particular interest for global FSBP operators, and highlight additional criteria that should be satisfied by the resulting operators and how these criteria can be incorporated in the construction. Second, I show how the optimization-based construction approach can be generalized to arbitrary point distributions in multiple dimensions. Finally, I present results for one- and two-dimensional numerical examples that demonstrate the superior performance and flexibility of the constructed FSBP operators compared to classical approaches.

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Mathematical model of current distribution in a tungsten plate during pulsed heating

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ABSTRACT

The report presents a new model of current distribution in a tungsten sample and an evaporated substance, taking into account the magnetic field during heating of the surface by an electron beam. The proposed mathematical model is a tool for non-destructive diagnostic systems for subsurface damage of various products by analyzing the reaction to rapid heating. The model is based on the solution of the equations of electrodynamics and the two-phase Stefan problem for calculating the temperature in the sample region in a cylindrical coordinate system. The model parameters are taken from experiments on the Beam of Electrons for Materials Test Applications (BETA) stand [1].

In 2024, a result was obtained [2] indicating that in the model under consideration, the maximum amplitude of thermal currents on the heated surface is outside the region of the rotating melt. In order to limit the region of currents, the effect of the magnetic field on the kinetics of electrons was added to the mathematical model. As a result, the conductivity and thermoelectric power take the form of a tensor. A new parameter is introduced - the ratio of the time between collisions to the period of rotation along the Larmor orbit. The coefficients are obtained from the generalized Ohm's law for plasma. The use of coefficients in a new form allowed to describe the distribution of currents in the melt. Since the plasma density rapidly decreases along the radius, the current for the region outside the radius in the plasma is suppressed. The mathematical model is supplemented by taking into account the Joule heating and the Peltier effect on the sample surface, which are determined from the calculation of currents [3]. The results of the modeling are required to analyze the data obtained at the experimental stand.

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Positivity-preserving Well-balanced PAMPA Scheme with Global Flux Quadrature for One-dimensional Shallow Water Equations

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Keywords: Well-balanced scheme; PAMPA method; high order methods; global flux positivity-preserving.

ABSTRACT

In this talk, we present a novel hydrostatic and non-hydrostatic equilibria preserving Point-Average-Moment Polynomial-interpreted (PAMPA) method for solving the one-dimensional shallow water equations. The idea is based on a global flux quadrature formulation, in which the discretization of the source terms is obtained from the derivative of an additional flux function computed via high order quadrature of the source term. The reformulated system is quasi-conservative with global integral terms computed using Gauss–Lobatto quadrature nodes. The resulting method is capable of preserving a large family of smooth moving equilibria: supercritical and subcritical flows, in a super-convergent manner. We also show that, by an appropriate quadrature strategy for the source, we can exactly preserve the still water states. Moreover, to guarantee the positivity of water depth and eliminate the spurious oscillations near shocks, we blend the high-order PAMPA schemes with the first order local Lax–Friedrichs schemes using the method developed in [1]. The first-order schemes are designed to preserve the still water equilibria and positivity of water height, as well as to deal with wet-dry fronts. Extensive numerical experiments are tested to validate the advantages and robustness of the proposed scheme.

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Structure-preserving schemes for symmetric hyperbolic and thermodynamically compatible systems of PDEs

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Keywords: structure preserving (SP) schemes, symmetric hyperbolic and thermodynamically compatible (SHTC) systems, hyperbolic thermodynamically compatible (HTC) finite volume schemes, semi-implicit discretization based on the symmetric Godunov form of the PDE, nonlinear acoustic equations, nonlinear Maxwell system, augmented nonlinear Maxwell-GLM system.

ABSTRACT

The proposed talk will present the development of exactly energy-conservative and structure-preserving finite volume schemes for the discretisation of first-order symmetric-hyperbolic and thermodynamically compatible (SHTC) systems of partial differential equations in continuum physics. Due to their thermodynamic compatibility the class of SHTC systems satisfy an additional conservation law for the total energy and many PDEs in this class of equations also satisfy stationary differential constraints (involutions). Consider, for example, the Maxwell-GLM system [1], which is composed of the original Maxwell equations augmented by the addition of two supplementary acoustic systems. The mathematical structure of the augmented Maxwell-GLM system is very intriguing, containing an interesting combination of curl-curl and div-grad operators with curl- and divergence-type constraints. This makes it a useful prototype system for developing new structure-preserving numerical methods.

In this direction, we will present a new fully discrete semi-implicit finite volume scheme that conserves total energy and which can be proven to satisfy also the involution constraints exactly at the discrete level. This method is a vertex-based staggered semi-implicit scheme that preserves the basic vector calculus identities $\nabla \cdot \nabla \times \mathbf{A} = 0$ and $\nabla \times \nabla \phi = 0$ for any vector and scalar field, respectively, exactly at the discrete level and which is also exactly totally energy conservative. The key feature of the proposed implicit scheme is its use of a discrete version of the *symmetric-hyperbolic Godunov-form* of the governing PDE system. This naturally leads to sequences of symmetric and positive definite linear algebraic systems that are solved using an iterative fixed-point method at each time step. We apply our new schemes to three different SHTC systems. Specifically, we consider i) the equations of nonlinear acoustics, ii) the nonlinear Maxwell equations in the absence of charges, and iii) a nonlinear version of the Maxwell-GLM system. We also present numerical results that demonstrate the stated properties of the proposed schemes.

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High-Order Methods for Euler Equations: Truly Multidimensional Operators and Selection Criteria

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Keywords: high-order finite volume schemes; Active Flux methods; truly multidimensional evolution operators; WENO reconstruction; physically relevant selections

ABSTRACT

We present a novel third-order fully-discrete Active Flux method based on the multidimensional approximate evolution operators. The latter are derived applying the theory of bicharacteristics. An approximate evolution operator is applied to evolve a numerical solution at the cell interfaces to be used in the flux evaluation. We study the influence of various approximate truly multidimensional evolution operators on the stability and accuracy of the whole Active Flux method applied to the linear wave equation system [1]. In particular, we discuss the influence of various third-order reconstructions in space on the stability and accuracy of the Active Flux method. We present the applications to the nonlinear Euler equations [1, 2]. We also discuss the recent results on rigorous convergence analysis via dissipative weak solutions derived for the first order variant of the method [3].

If time permits, we will also discuss the effects of higher-order approximations on the physically relevant selections of multidimensional Euler equations [4, 5].

The results on the Active Flux methods have been obtained in the collaboration with S. Chu, E. Chudzik, C. Helzel, A. Porfetye, and Z. Tang. The results on the selections and higher-order schemes have been obtained in the collaboration with S. Chu, E. Feireisl, M. Herty, A. Kurganov, and C. Yu.

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Revisiting Genuinely Multidimensional Fluxes for Cell-Centered Lagrangian Gas Dynamics

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Keywords: finite volume schemes; Lagrangian methods.

ABSTRACT

The Lagrangian kinematic representation is particularly well suited not only for tracking shock and compression waves, but also for exactly capturing interfaces between distinct materials and for accurately following moving computational domains undergoing large deformations. Among the available numerical approaches [3], the cell-centered Finite Volume discretization of the Lagrangian gas dynamics equations is based on the direct discretization of the conservation laws of volume, momentum, and total energy on a mesh moving with the fluid velocity.

Within this framework, two key ingredients are required. First, a consistent numerical approximation of the interfacial fluxes must be defined to update the cell-centered states. Second, an approximation of the nodal velocity must be constructed to move the mesh in a manner compatible with the discrete evolution of the volume equation. In addition, the numerical fluxes are designed to satisfy a semi-discrete entropy inequality, ensuring the physically consistent conversion of kinetic energy into internal energy in accordance with the second law of thermodynamics. The construction of a robust Lagrangian gas dynamics discretization therefore relies on **genuinely multidimensional** flux and nodal velocity approximations; see [3] for representative examples.

In the present work, we revisit the cell-centered Finite Volume discretization of Lagrangian gas dynamics by exploiting the subface-based Finite Volume formalism introduced in [2] for the Eulerian gas dynamics equations. This approach leads to a novel positivity-preserving cell-centered Finite Volume discretization of the multidimensional Lagrangian gas dynamics, based on a partition of each cell face into subfaces impinging at the nodes [2].

The subface flux approximation is derived from an approximate Riemann solver that incorporates not only the mean values of the cells adjacent to the subface, but also the velocity of the node from which the subface emanates. The projection of the nodal velocity onto the unit normal vector of the subface may be interpreted as a parameter in this Riemann solver. As a consequence, the resulting subface flux is not uniquely defined, leading to a loss of conservation in the classical sense. Conservation is recovered by enforcing that the sum of subface fluxes around each node vanishes, which determines the nodal velocity through a **nodal solver**.

This node-based conservation condition can be reformulated as a sum of numerical fluctuations, thereby revealing a natural connection with the consistency condition used in the Residual Distribution framework [1]. The resulting scheme can thus be interpreted as a **multipoint flux approximation**, providing a formal reinterpretation of the original cell-centered Finite Volume formulation introduced more than twenty years ago [3]. This reinterpretation furthermore supplies a theoretical framework for establishing the positivity-preserving and entropy-stable properties of the proposed discretization.

We complement this analysis by considering an alternative cell-centered Finite Volume discretization in which the numerical fluxes are uniquely defined at the nodes [4]. By construction, this formulation is

conservative. Exploiting its fluctuation form, we derive a node-based condition linking the nodal fluxes in such a way as to satisfy a discrete entropy inequality. We then propose a genuinely multidimensional closure for the nodal fluxes that enforces this entropy condition. These nodal fluxes consist of a centered contribution, interpretable as a node-based average of the physical flux, augmented by a discrete operator of divergence/gradient type. The resulting multidimensional discretization is closely related to the formulation obtained by a different route in [5].

Finally, we compare the structure and properties of the aforementioned discretizations and evaluate their robustness and accuracy on a set of representative multidimensional test cases.

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High-order Minimally Implicit Runge-Kutta (MIRK) methods and application to the 1D shallow water equations with topography and Manning friction

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Keywords: systems of balance laws; stiff source terms; semi-implicit methods; minimally implicit methods; shallow water equations; high-order methods.

ABSTRACT

The Minimally Implicit Runge–Kutta (MIRK) methods are proposed as an efficient time-integration strategy for nonlinear hyperbolic balance laws with stiff source terms. Although Implicit–Explicit (IMEX) methods [1] are widely used for such problems, their computational efficiency reduces very significantly for high-order schemes due to the increasing number of implicit evaluations to be inverted. MIRK schemes overcome this limitation by retaining the stability advantages of semi-implicit approaches while drastically reducing the implicit coupling, leading to a substantially lower computational cost without sacrificing accuracy or robustness. The MIRK framework has previously been successfully applied in some astrophysical scenarios [2, 3], and for the one-dimensional shallow water equations with non-flat topography and Manning friction (1DSWF) [4], achieving first- and second-order accuracy. This contribution focuses on the theoretical formulation of extending MIRK methods to higher-order schemes. We also check their numerical performance in the 1DSWF, in terms of accuracy, stability, and computational efficiency compared to IMEX schemes.

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About the DoD Stabilization to Solve the Small Cell Problem

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Keywords: discontinuous Galerkin schemes; Cartesian cut cell mesh; small cell problem; error estimate; high order methods.

ABSTRACT

A by now popular approach for handling complex geometries in flow simulations is the use of *cut cell* meshes: the geometry is cut out of a structured background mesh, resulting in *cut cells* where the object intersects the mesh. When solving hyperbolic conservation laws, this causes the *small cell problem*: explicit time-stepping schemes become unstable unless the time step is restricted according to the smallest cut-cell volumes. Since these volumes can be arbitrarily small, enforcing such a constraint leads to prohibitively small time steps and is therefore impractical in realistic simulations.

The *Domain-of-Dependence* (DoD) stabilization introduced in [1] addresses this issue within the framework of discontinuous Galerkin (DG) schemes. The approach augments the spatial discretization with carefully designed penalty terms that directly couple the inflow and outflow neighbors of small cut cells. As a result, standard explicit time-stepping schemes can be employed without additional restrictions: the time step size can be selected according to the background mesh resolution rather than the potentially vanishing size of the cut cells.

In this talk, we introduce the cut cell approach and the small cell problem in more detail to provide a deeper understanding. We then present the ideas behind the DoD stabilization and discuss recent developments. This includes an error estimate for the DoD stabilized scheme [2] as well as new extensions to the two-dimensional wave equation [3].

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All Mach number second-order implicit-explicit schemes for isentropic compressible Cahn-Hilliard-Navier-Stokes equations

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Keywords: Asymptotic preserving; low Mach number; implicit-explicit schemes; incompressible limit; isentropic compressible Cahn-Hilliard-Navier-Stokes equations

ABSTRACT

We propose a second-order all Mach number implicit-explicit (IMEX) time-stepping scheme for the isentropic, compressible Cahn-Hilliard-Navier-Stokes equations discretized on staggered (MAC) grids. The scheme is based on finite difference approximations that ensure a stable coupling among the velocity, density and phase field, with symmetric operators acting on the discretized viscosity terms.

Standard explicit methods suffer from severe time-step restrictions due to the presence of second to fourth-order diffusion terms introduced by the Cahn-Hilliard and Navier-Stokes operators. To overcome these challenges, we develop a second-order IMEX Runge-Kutta scheme that treats the stiff terms, including part of the pressure, implicitly while the others are dealt with explicitly.

Numerical experiments are performed to verify the stability, accuracy and efficiency of the proposed approach.

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A generalized Riemann problem-based compact reconstruction method for finite volume schemes

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Keywords: Reconstruction methods; generalized Riemann problems; Godunov's theorem.

ABSTRACT

We present a Generalized Riemann Problem-based reconstruction method (GRPrec) for high-order finite volume schemes applied to hyperbolic partial differential equations. The method constructs spatial polynomials using cell averages at the current time level and GRP solution data from the previous time level. The resulting GRPrec stencil is as compact as that of discontinuous Galerkin (DG) schemes but unlike DG, our finite volume schemes obey a generous CFL stability condition that is independent of the order of accuracy. We assess the method's performance through test problems for smooth and discontinuous solutions of the linear advection equation and the Euler equations of gas dynamics in one space dimension. Results are compared against exact solutions and against numerical results from well-known spatial reconstruction finite volume and DG schemes, with all methods implemented in the fully discrete ADER framework. The performance of GRPrec is very promising, especially in terms of efficiency, that is error against CPU cost.

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A Hyperbolic Approximation of the Nonlinear Schrödinger Equation

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Keywords: hyperbolic approximation; nonlinear Schrödinger equation; finite volume schemes; implicit-explicit methods; Hamiltonian structure; special solutions.

ABSTRACT

We study a first-order hyperbolic approximation of the nonlinear Schrödinger (NLS) equation in one and higher spatial dimensions. The proposed hyperbolic system is shown to be strictly hyperbolic and to possess a modified Hamiltonian structure, along with conserved quantities that approximate the mass and energy of NLS. We establish the existence of standing-wave solutions (explicit in the one-dimensional case) that are closely related to ground-state solutions of NLS. Moreover, we show that solutions of the hyperbolic system converge to those of NLS in the relaxation limit under appropriate regularity assumptions.

We develop high-order asymptotic-preserving numerical discretizations that converge to consistent discretizations of NLS while exactly conserving mass. Furthermore, a robust and fully explicit finite volume scheme for the hyperbolic system is proposed. This discretization allows us to use patch-based adaptive mesh refinement on logically rectangular mapped grids with non-periodic boundary conditions. Numerical experiments in both the focusing and defocusing regimes demonstrate the accuracy and robustness of the approach, with applications including Bose–Einstein condensates, line-soliton instabilities, and superflow past obstacles (see Figure 1).

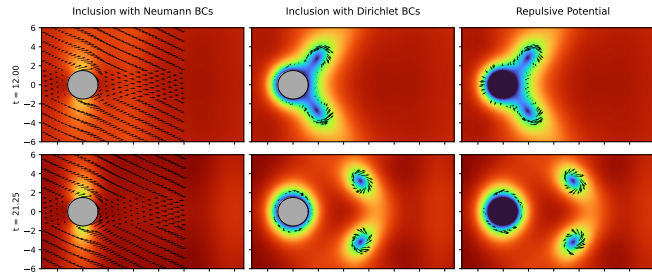


Figure 1: Superflow past a circular inclusion using different boundary conditions for our hyperbolic approximation of the defocusing NLS equation

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An APITALI approach for Lagrangian schemes enforcing the discrete entropy growth for Euler equations

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Keywords: Finite-Volume schemes; Lagrangian methods; High-Order methods.

ABSTRACT

High-order numerical schemes are increasingly used for simulating compressible flows due to their ability to capture fine-scale features with high accuracy. However, ensuring physical fidelity in these schemes, particularly in the presence of shocks and discontinuities, remains a significant challenge. One critical aspect is the discrete entropy growth which is essential for maintaining the thermodynamic consistency of numerical solutions. Entropy is the nonlinear stability criterion for Euler equations and is generally not used in MOOD and APITALI methods. In this study, we explore the application of the APITALI (A Posteriori ITerActive Limiter) method to address entropy variation issues in the GAIA scheme (see [3]) for solving Euler equations in pure Lagrangian coordinates, making our method invariant-domain preserving.

The APITALI method (see [2]) is an a posteriori limiter method. In this work, we use it to enforce entropy stability by adaptively adjusting high-order fluxes based on local entropy growth. This method applies to incomplete equations of state (EOS) by estimating the variation of entropy. The validity of this approach is verified with comparison to complete EOS calculation. This strategy ensures compliance with the second law of thermodynamics and prevents spurious oscillations near discontinuities.

We validate our approach using the challenging Bizarrium test case (see [1]), characterized by non-convex isentropes and complex shock structures. Without APITALI, high-order schemes such as GAIA with TVD or Minmod limiters fail to converge or capture the physical solution. With APITALI the schemes converge robustly, accurately capturing the entropic solution. These results highlight the importance to carefully tune numerical dissipation to achieve a balance between stability and accuracy.

We also apply the method to multi-dimensional Lagrangian schemes as high-order extensions of Glace and Eucclhyd. Relevant test problems show that APITALI also performs well in this framework. This work demonstrates the effectiveness of the APITALI method in ensuring entropy stability for high-order Lagrangian schemes, particularly for non-convex EOS and complex shock interactions.

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Boltzmann Across Regimes: Dynamic Domain Decomposition and Uniformly Accurate IMEX Solvers

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ABSTRACT

The efficient simulation of non-equilibrium flows requires a strategic balance between physical fidelity and computational overhead. In this talk, we present a comprehensive framework for multiscale kinetic simulations based on a dynamic hierarchy of models, ranging from the full Boltzmann equation and the ES-BGK approximation to the macroscopic Euler equations. The cornerstone of our methodology is a dynamic domain decomposition strategy that adaptively assigns the optimal model to different regions of the phase space and physical domain. This multi-level partitioning is seamlessly integrated with high-order aligned Implicit-Explicit (IMEX) Runge-Kutta methods, ensuring uniform high order accuracy across all regimes. We further discuss the challenges of extending such a hierarchical approach on unstructured grids, providing the necessary geometric flexibility for complex, real-world applications. This is a joint research with D. Caparello and T. Rey.

Discrete Hodge-Helmholtz decomposition and application to the accuracy problem in the computation of low Mach number compressible flows.

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Keywords: finite volume schemes; discontinuous Galerkin finite element schemes; discrete exterior calculus;

ABSTRACT

This talk will be dedicated to the continuous and discrete aspects of Hodge-Helmholtz decomposition, i.e., the decomposition of a vector field into the sum of a divergence free and a curl free component. From a continuous point of view, we will see how it can be used, for example, to characterize the long time limit of the wave system. We will then present a discrete decomposition of piecewise constant vector fields on triangles, proposed in [2], and adapted to the analysis of finite volume schemes for the wave system [3, 4].

Then, we will recast this decomposition for finite volumes on triangles in the context of discrete exterior calculus [1], and in particular in the context of discrete distributional de Rham complexes, proposed in [6]. We will see how to extend these diagrams to approximation spaces compatible with high-order methods of the discontinuous Galerkin type, and how to extend the results from triangular meshes to quadrangular meshes.

Based on these approximation spaces, we will see how to develop numerical methods that preserve divergence or curl constraints (e.g., MHD or Maxwell), and how to solve the accuracy problem of finite volume/discontinuous Galerkin schemes at low Mach numbers.

Most of the work presented in this talk was obtained in collaboration with Jonathan Jung (UPPA) and was published in [4, 5, 8, 7].

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On various structures of continuum mechanics models and the need for structure preserving integrators

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Keywords: continuum mechanics; differential identities; Hamiltonian and variational structures.

ABSTRACT

In this talk, we will discuss various structures of continuum mechanics models such as Hamiltonian and variational structures, Godunov structure, differential geometric structures, etc. As examples, we will consider the Maxwell equations, unified model of continuum mechanics, hyperbolic heat equation, etc. All these features make any model overdetermined and impose extra challenges for designing numerical methods that goes beyond the usual task of discretizing the balance laws of a model.

SCOUT: Semi-Lagrangian Conservative and Unconditionally sTable schemes for nonlinear advection-diffusion problems

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Keywords: semi-Lagrangian schemes; conservation; unconditional stability; advection-diffusion equation; nonlinear PDE.

ABSTRACT

In this talk, we present a new conservative semi-Lagrangian finite difference scheme for nonlinear advection-diffusion problems. Conservation, which is fundamental for obtaining physically consistent solutions, is ensured by integrating the governing equations over a space-time control volume constructed along the characteristic curves originating from each computational point. The subsequent application of Gauss theorem allows the evaluation of all space-time surface integrals and enables the computation of the time integrals of the flux using only spatial information, which is available at the beginning of the timestep. For nonlinear problems, a nonlinear equation must be solved to determine the foot of the characteristic, whereas this step is not needed in linear cases. Furthermore, we show how the diffusion terms can be directly incorporated within the same framework, leading to the development of a novel characteristic-based Crank-Nicolson discretization in which the diffusion contribution is integrated along the characteristic. A broad set of benchmark tests is presented to assess the accuracy, robustness, and strict conservation property of the proposed method, as well as its unconditional stability, which is verified by numerical experiments with CFL numbers up to 100.

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From kinetic equations to macroscopic models for two phase flow

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Keywords: finite volume schemes; discontinuous Galerkin finite element schemes; Lagrangian methods; Eulerian methods; ALE methods; high order methods.

ABSTRACT

This talk addresses the problem of deriving macroscopic models from microscopic dynamics. Many models we discretize to produce numerical solutions derive from Boltzmann equation. The derivation from kinetic equations enriches a model with the possibility of computing transport coefficients, provides automatically entropy dissipation, or embeds the macroscopic equations into a more general setting in which the macroscopic equation is obtained when the system is in equilibrium. Much effort is spent in deriving macroscopic models for mixtures of fluids from kinetic models for mixtures. Models derived in this fashion are very effective to derive equations for plasmas. But this approach has so far failed to derive the equations currently used for multiphase flows from Boltzmann equations for mixtures. In this talk, we propose a new approach which, we think, can bridge this gap.

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High-Order Positivity-Preserving Well-Balanced Schemes: Combining Flux Globalization with Modified Patankar Methods

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Keywords: high-order methods, finite volume schemes, well-balanced schemes, Modified Patankar methods, shallow water flows

ABSTRACT

Modified Patankar schemes are time-advancing methods for solutions to production-destruction systems. Through nonlinear weighting, these schemes unconditionally ensure conservation and positivity within these systems. In the context of hyperbolic conservation laws, this also proves fruitful, as these methods can be used to ensure positivity of physical quantities such as density or fluid height. Furthermore, the semi-discrete conservative form of many finite volume schemes fit quite naturally into the production-destruction-type setup required for modified Patankar. This fairly seamless combination extends to the quasi-conservative setup of hyperbolic balance laws, and, with the help of existing flux-globalization techniques, provides a simple way to ensure both positivity and an exact preservation of static and moving equilibria. In this work, we present a high-order, positivity-preserving, flux-globalization-based well-balanced scheme for the shallow water system, which is discretized in time by the arbitrary-order modified Patankar deferred correction method. Through numerical experiments, we validate the effectiveness of this approach in capturing the multiscale phenomena and ensuring positivity of the fluid height, demonstrating the potential of such methods in real applications such as tsunami simulations.

Robust discontinuous Galerkin solution of the multicomponent Euler model for multiphase flows

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Keywords: discontinuous Galerkin, multicomponent Euler model, multiphase flows, stiffened gas, robustness

ABSTRACT

This talk presents the results obtained with a modal discontinuous Galerkin (dG) solver for the multicomponent Euler equations closed with the stiffened gas equation of state [1]. High-order methods such as dG produce nonphysical values at flow discontinuities within cells, due to Gibbs phenomena and possible pressure oscillations at material interfaces. An increasingly popular *a posteriori* mitigation technique is to re-compute the solution of a “troubled” element by mapping the solution at the previous time step into smaller sub-cells, and obtain the new solution using a more robust scheme in this subdomain [2]. The talk will present the preliminary results obtained with this technique and compare it with an artificial-viscosity shock-capturing approach [3], focusing on how each strategy stabilize numerically diffuse liquid–gas interfaces. We include the comparison of different sensors used in both methods, critical for detecting the zones at risk of producing non-physical solutions. Furthermore, the differences between the high-order strategies with respect to a MOOD-like limiter [4] are presented. We assess the robustness of the scheme by simulating a series of Riemann problems of increasing difficulty, together with a two-dimensional shock-drop interaction.

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Genuinely multidimensional Riemann solvers, residual distribution and continuous finite elements: a unified view with some recent results.

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ABSTRACT

We are interested in the design of genuinely multidimensional numerical schemes for nonlinear hyperbolic conservation laws systems. The term “genuinely multidimensional” here refers to several aspects:

- the methods in question must not, in principle, be reduced to the application of 1D schemes in directions dictated by the mesh
- the methods in question must not, in principle, use the same simple geometric notion of upwinding that comes from 1D
- numerical dissipation must not be based on differences but on multidimensional differential operators (div or grad at points or in the mesh).

This talk will highlight the links between schemes from the world of stabilized continuous finite elements (FE) , residual distribution schemes (RD), and multidimensional Riemann solvers with point fluxes. This bridge allows to propose schemes mixing ideas from these different worlds. Some examples will be provided.

Arbitrary-Lagrangian-Eulerian solver for relativistic detonation waves

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Keywords: Shock waves; Relativistic Euler equations; Reaction flows; Lagrangian-Eulerian scheme; Zeldovich spike.

ABSTRACT

In this work we present a one dimensional Lagrangian code for relativistic detonations. The problem is described by the equation for the conservation of mass and of energy-momentum, with an extra evolution equation that governs the variation of the reaction rate along the fluid flow. These are analyzed in a one dimensional setting. The shock wave profile is achieved by integrating the PDEs in rest frame of the shock with an Arbitrary Lagrangian Eulerian scheme [1]. This solution is validated with the one obtained through the integration of the associated ODE system, which is reached by performing a Lorentz transformation into the rest frame of the shock front. In the one dimensional setting it is possible to derive the Rankine Hugoniot conditions. These can be used to obtain analytical equations for each variable in relation to the one representing the reaction rate. The ODE system is then reduced to a single ODE for the reaction rate. This ODE is integrated with a well known Runge Kutta method.

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Numerical Evaluation of Effective Elastic Moduli of a Heterogeneous Deformed Fluid Saturated Porous Medium

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Keywords: Effective elastic moduli; saturated porous medium; strain energy-based method; time marching method for steady-state solution; hyperbolic thermodynamically compatible model.

ABSTRACT

A method for evaluation the effective elastic moduli for fluid-saturated deformable porous materials is proposed. The method is based on calculating the strain energy of an heterogeneous medium and comparing it with the energy of an equivalent homogeneous elastic medium. The heterogeneous medium is modeled using a Hyperbolic Thermodynamically Compatible (HTC) system of equations of a porous medium [1] simplified for small deformations [2], which allows the use of a diffuse interface approach in the presence of regions of a purely elastic medium or purely fluid. Comparing the energies of a porous and equivalent elastic medium while solving a series of simple stationary problems on the stress-strain state of a sample allows us to calculate all components of the effective compliance tensor, and therefore all components of the effective stiffness tensor, as well as the effective sound velocities. For the numerical solution of the governing equations, the high-order finite difference method on a staggered grid can be effectively applied. To solve stationary problems on the equilibrium of a sample, a steady-state solution is obtained using the time-marching method with artificial velocity relaxation.

A series of test problems are presented demonstrating the flexibility of the method for application to various types of saturated porous medium samples. A comparison is made with known theoretical predictions of effective moduli for certain media. It is shown, in particular, that for a sample of a liquid-saturated porous elastic medium of the simplest periodic structure, the elastic moduli used in the HTC model are approximated with good accuracy by the numerically obtained effective moduli.

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Asymptotic Preserving and Well Balanced Linearly Implicit Additive IMEX Schemes for the Isentropic Euler Equations with Gravity

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Keywords: Finite volume schemes; Euler Equations with Gravity; Asymptotic Preserving; IMEX Schemes; Well-balanced.

ABSTRACT

Numerical approximation of the compressible Euler equations is a challenging task owing to the multi-scale nature of the flow dynamics. We consider the isentropic Euler equations with source terms modeling the effect of a gravitational potential, under anelastic scaling. In the limit of zero Mach number, the system reduces to a variable-density incompressible Euler system. Although implicit or semi-implicit schemes are clear front-runners for obtaining an asymptotic-preserving (AP) scheme that not only captures the limiting behavior but also remains stable, the nonlinearity of the equation of state makes their numerical resolution cumbersome. Moreover, the system admits steady solutions, and it is well known that a robust numerical scheme must preserve these states, as most of the phenomena under study are small perturbations of these steady states.

We present the development and analysis of a higher-order semi-implicit finite volume scheme for the governing equations under consideration that is both AP and well-balanced. The time integration hinges on the identification of an equivalent linear steady state and a penalization approach [1], to employ additively split IMEX–RK [2] schemes. This yields a linearly semi-implicit scheme. Although the IMEX approach leads to a formally AP scheme, maintaining balance for all values of the Mach number remains a challenge. This issue is addressed through a combination of appropriately chosen reconstruction variables and a carefully designed source-term discretization. Numerical case studies carried out corroborate the claimed theoretical properties.

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A Structure Preserving Finite Volume Scheme for the Navier-Stokes-Korteweg Equations

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Keywords: Navier-Stokes-Korteweg, Finite Volume, Structure Preserving, Fallback Method, Energy Dissipation.

ABSTRACT

The Navier-Stokes-Korteweg (NSK) system of equations extends the Navier-Stokes system to model capillarity effects in fluids. We propose a simple finite volume method to numerically solve the NSK system on an equidistant cartesian grid with periodic boundary conditions which has a novel combination of properties: It conserves mass and momentum, is energy dissipative, and operates directly on the NSK system without employing hyperbolic approximation or auxiliary variables. The proposed method is intended as a fallback scheme for structure preserving higher order schemes that rely on limiters [3] or on the MOOD approach [1]. Possible extensions to higher order will be discussed. We present numerical experiments to validate our claims.

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High-Order Shock Capturing on Heterogeneous Grids using Discontinuous Galerkin Spectral Element Methods

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Keywords: finite volume schemes; high order methods; discontinuous Galerkin scheme; heterogeneous grids;

ABSTRACT

The discontinuous Galerkin method allows a straightforward construction of efficient high-order schemes on arbitrary grids for hyperbolic partial differential equations (PDEs) such as the Euler equations. However, hyperbolic PDEs admit discontinuities in the solution, even if the initial solution is smooth. Moreover, it is well known that high-order operators induce oscillations at discontinuities, called Gibbs phenomena. Hence, adequate numerical treatment is necessary to detect and handle discontinuities. One example is a switching procedure where a troubled DG cell is switched to a low-order finite volume scheme, where the latter is defined on an equidistant point set. This approach enables a highly accurate and localized shock capturing. However, implementations of h-adaptive FV subcell schemes for DGSEM and related DG schemes were so far limited to hexahedral and tetrahedral elements.

In this talk, a finite volume subcell shock capturing procedure for discontinuous Galerkin spectral element method on heterogeneous grids, building on the work presented in [1], is proposed. The proposed scheme is based on an h-adaptive second-order total variational diminishing finite volume operator defined on an equidistant point set. The capabilities of the shock capturing procedure are demonstrated on a series of well-known numerical experiments including an application-oriented numerical problem.

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Superconvergence through SIAC filtering on non-uniform, structured meshes

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Keywords: post-processing; SIAC-filtering; finite elements; discontinuous Galerkin finite element schemes; high order methods

ABSTRACT

Smoothness Increasing Accuracy Conserving (SIAC) filters are convolution kernels which are traditionally used as a post-processing tool to reduce the error, increase the smoothness and enhance the convergence rate of finite element approximations of Partial Differential Equations (PDE).

They have been extensively analyzed in the setting of discontinuous Galerkin methods for hyperbolic conservation laws. In this context, their application can improve the order of accuracy on translation-invariant meshes from $p + 1$ to $2p + 1$ for polynomials of degree p . This even holds for nonlinear hyperbolic problems, in regions where the solution remains smooth.

However, the performance of the post-processing and its ability to recover superconvergence is strongly dependent on the mesh uniformity. While existing strategies for SIAC filtering on nonuniform meshes can lead to superconvergence of order $2p + 1$ on smoothly varying meshes, the rate of superconvergence is a priori unknown and might drop to $p + 1$, on randomly perturbed structured meshes.

We propose a novel strategy to apply SIAC filtering on nonuniform but structured meshes. The approach is based on the idea of transforming the structured mesh to a translation invariant one, where a SIAC filter is applied. The transformation is required to be a smooth diffeomorphism to ensure the rate improvement. We provide a theoretical justification for the approach, discuss the assumptions and provide a method to construct the transformation.

The transformation eliminates the need to identify integration regions for individual physical integration points, thereby simplifying the pre-processing work and enabling uniform reuse of local post-processing operations in the interior and the boundary. We provide numerical examples to demonstrate the performance of the approach, suggesting that by using an appropriate transformation, superconvergence of order $2p + 1$ can be recovered.

A structure-preserving semi-implicit four-split scheme for continuum mechanics

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Keywords: structure preserving; asymptotic-preserving; semi-implicit methods; staggered meshes; unified model of continuum mechanics.

ABSTRACT

In this contribution, we introduce a novel semi-implicit four-split discretization of a unified first order hyperbolic formulation of continuum mechanics [1]. The governing PDE system goes back to the pioneering work of Godunov, Romenski, Peshkov and collaborators. This model, which was first solved numerically in [2], allows to recover both the fluid and the solid regimes as the limits of a characteristic relaxation time. Previous structure-preserving discretizations of this system allowed to respect the curl-free properties of the distortion field and of the specific thermal impulse in the absence of source terms and were also able to properly deal with the low Mach number limit with respect to the adiabatic sound speed. However, since the shear wave speed may be large in the solid limit, this still requires a rather severe CFL stability restriction. In the proposed approach, the resulting CFL time step is limited only by the maximum velocity magnitude and not by the shear, pressure or heat waves speeds. For this purpose, the governing PDE system is split into four subsystems: i) a convective subsystem, which is the only one that is treated explicitly; ii) a heat subsystem, iii) a subsystem that embeds momentum, distortion field and specific thermal impulse; iv) a pressure subsystem. To relax the CFL stability restrictions, the last three subsystems are discretized implicitly. The method is consistent with the low Mach number limit of the equations, and the Navier-Stokes viscous stress tensor can formally be recovered at the analytical level in the fluid limit. We show several numerical results for classical benchmark problems that allow to assess the performance of the scheme in different asymptotic limits of the governing equations, including the fluid and solid limit.

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Numerical study of some problems of rarefied gas mechanics

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ABSTRACT

In recent years, we have been developing numerical methods for solving kinetic equations with approximate (model) collision operators of the BGK and E.M. Shakhov models, as applied to complex multidimensional flows of a monatomic rarefied gas. These methods are being implemented in the parallel computational code "Nesvetay" [1, 2]. The current talk discusses the application of these developments to solving two problems that can be included in the database of benchmark problems in computational mechanics of rarefied gases.

The first problem simulates the expansion of a gas cloud into a vacuum due to evaporation from a solid surface, induced by nanosecond laser irradiation of moderate intensity. The goal of the calculations is to construct a flow pattern, as well as to obtain the time-of-flight and angular distributions of the expanding particles, which requires high-precision integration of the distribution function in velocity space. To verify the solution, a comparison is made with the LasInEx (Laser Induced Expansion) calculation code, developed by the second author and based on the direct simulation Monte Carlo (DSMC) method [2]. Good quantitative and qualitative agreement of the results was obtained for all calculated quantities (field, time-of-flight distribution function, average molecular energy) for moderate and strong evaporation regimes.

The second problem examines the flow of a rarefied gas under a pressure difference through a periodic system of short channels, simulating a plane membrane. The primary objective is to study the influence of surface temperature and degree of rarefaction on the mass flow rate and flow pattern as opposed to the case of the surface temperature, equal to the inflow gas temperature [3]. The analysis is based on both kinetic modeling and the Navier-Stokes equations. It is shown that the effect of surface temperature on gas flow rate is significant in rarefied and transient flow regimes and weakens as a continuous medium is approached. Moreover, the dependence of gas flow rate on surface temperature is close to linear, which is favorable for applications. For all values of surface temperature, arc-shaped shock waves appear in the flow for regimes close to the continuum.

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Numerical methods for hyperbolic multi-scale systems

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ABSTRACT

Multi-scale problems are omnipresent in environmental and industrial processes, posing a challenge to classical numerical solvers given that the propagation speeds of information span several orders of magnitude. In hyperbolic systems, the absolute fastest wave speed remains finite, but in classical, explicitly integrated numerical schemes, it determines the time step that ensures stability. This means in particular that it may lead to vanishing time increments in the presence of fast processes caused for instance by high pressure or strong magnetic fields. Furthermore, it is known that upwind schemes introduce spurious numerical diffusion into the approximate solution, a problem that can only be partially mitigated by mesh refinement. Therefore, a common approach, which will be also applied here, is to use implicit or semi-implicit time integrators, combined with a suitable approximation of spatial derivatives in the implicitly treated systems. This allows scale-independent artificial dissipation and stability to be obtained under large time steps. As the evolution of the modeled variables is described by a nonlinear flux function, treating it fully or partially implicitly involves solving nonlinear systems. Depending on the problem, these systems can be large, coupled and ill-posed, for which solvers such as the Newton method may converge very slowly or not at all. To avoid nonlinear implicit systems, we apply a subtle linearization while keeping the systems compatible with crucial properties such as asymptotic limits, conservation of mass, momentum and energy, and necessary vector-calculus identities. In this talk, the construction of such numerical schemes in the finite volume framework will be illustrated at the example of a model of continuum mechanics consisting of at least three scales including a convective and an acoustic sub-system.

Time-reconstruction solvers for the GRPm. Application to ADER methods for hyperbolic systems.

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ABSTRACT

The generalized Riemann problem with piece-wise smooth polynomial data of degree m (GRPm) is a Cauchy problem of intrinsic interest for inhomogeneous hyperbolic systems. The solution at the interface consists of a polynomial in time of degree m , which could be obtained in semi-analytical form or fully computational form, depending on the particular GRPm solver in use. A practical use of such interface time polynomial is the computation of the numerical flux in ADER methods, which produces schemes of order $m + 1$ in space and time, for arbitrary m .

In this talk I will first briefly list existing solvers for GRPm. Then, I will describe in some detail, a time-reconstruction (TR) approach, still in development. The TR method uses old data to reconstruct the time polynomial, avoids the computationally-intensive Cauchy-Kovalevskaya procedure and, potentially, can deal with stiff source terms, while preserving high accuracy. Numerical results for multidimensional problems will be shown.

An Element-Based Convex Limiting Framework for Continuous Galerkin Methods with Nonlinear Stabilization

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Keywords: invariant domain preservation; high-order finite elements; WENO stabilization; monolithic convex limiting.

ABSTRACT

We equip a high-order continuous Galerkin discretization of a general hyperbolic problem with a nonlinear stabilization term and introduce a new methodology for enforcing preservation of invariant domains. The amount of shock-capturing artificial viscosity is determined by a smoothness sensor that measures deviations from a weighted essentially nonoscillatory (WENO) reconstruction. Since this kind of dissipative stabilization does not guarantee that the nodal states of the finite element approximation stay in a convex admissible set, we adaptively constrain deviations of these states from intermediate cell averages. The representation of our scheme in terms of such cell averages makes it possible to apply convex limiting techniques originally designed for positivity-preserving discontinuous Galerkin (DG) methods. Adapting these techniques to the continuous Galerkin setting and using Bernstein polynomials as local basis functions, we prove the invariant domain preservation property under a time step restriction that can be significantly weakened by using a flux limiter for the auxiliary cell averages. The close relationship to DG-WENO schemes is exploited and discussed. All algorithmic steps can be implemented in a matrix-free and hardware-aware manner. The effectiveness of the new element-based limiting strategy is illustrated by numerical examples.

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Subcell monolithic ADER-AFE-DG/FV convex property preserving scheme

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Keywords: Discontinuous Galerkin schemes; Finite Volume schemes; monolithic schemes; Polygonal grids; Agglomerated Finite Element basis; Subcell reformulation; High-order methods.

ABSTRACT

This talk aims at presenting a new monolithic convex property preserving scheme, blending at the subcell level, a high-order ADER nodal Discontinuous Galerkin (DG) scheme defined on unstructured polygonal meshes with a first-order Finite Volume (FV) method defined on a subgrid. Regarding the nodal DG scheme, [1], rather than using classical polynomials inside each element, in this new approach the discrete solution is represented by piecewise continuous polynomials within each Voronoi element, using a continuous finite element basis defined on sub-triangles inside each polygon, see Figure 1(a). This polygonal nodal DG is thus referred to as Agglomerated Finite Element DG (AFE-DG).

However, as any high-order scheme, this method needs some sort of nonlinear stabilization. To do so, we use the approach recently introduced in [2]. The goal here is to blend, at a way smaller scale than the cell, the high-order scheme with a more robust one. To this end, we first show that it is possible to rewrite ADER-AFE-DG scheme as a one step FV-like method defined on a subgrid, see Figure 1(b), provided with some specific fluxes referred to as reconstructed fluxes. The reconstructed fluxes are then blended, as the subcell scale, with a 1st-order FV numerical flux. The goal is now to determine, through analysis, optimal blending coefficients to achieve the desired properties. Numerical results on various type problems will be presented to assess the very good performance of the design method.

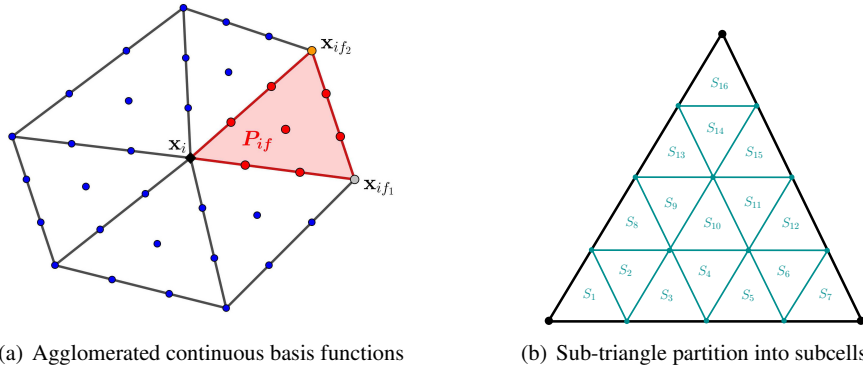


Figure 1: Degrees of freedom on the polygonal cell and subdivision of the sub-triangles into subcells.

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A semi-Lagrangian method for the polyatomic ESBGK model

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Keywords: kinetic equation; polyatomic ESBGK; semi-Lagrangian method.

ABSTRACT

Polyatomic kinetic models are essential for accurately capturing the thermodynamic behavior of real gases, as internal energy modes significantly influence transport coefficients, relaxation processes, and non-equilibrium effects that cannot be represented by monoatomic models. The polyatomic ESBGK model [1] describes molecular collisions as a relaxation towards a generalized Gaussian distribution with an anisotropic covariance matrix and an exponentially decaying internal energy distribution. We present a new semi-Lagrangian scheme for the polyatomic Ellipsoidal Statistical BGK (ESBGK) model of the Boltzmann equation.

The semi-Lagrangian framework [2], being deterministic and grid-based, removes the time-step restriction associated with the linear transport term by following the method of characteristics. The potentially stiff relaxation term is treated using an implicit A-stable linear multistep method which, owing to the structure of the BGK operator, can be reformulated into a fully explicit time-stepping scheme. This yields a highly efficient and numerically stable method. Ultimately, the method is second-order in time, third-order in space, and spectral in the microscopic velocity. The numerical method is asymptotic preserving and stiffly accurate, meaning the scheme asymptotically converges to a scheme for the Euler equations in the vanishing Knudsen limit. In addition, we prove that the first-order scheme, asymptotically converges to the compressible Navier-Stokes equation. Finally, we propose inflow and outflow boundary conditions suitable for BGK-type kinetic equations. We perform simulations of the Fourier and Couette test case to compare the BGK model with Direct Simulation Monte Carlo (DSMC). In addition, we perform simulations of an orifice geometry.

Fourier Flow

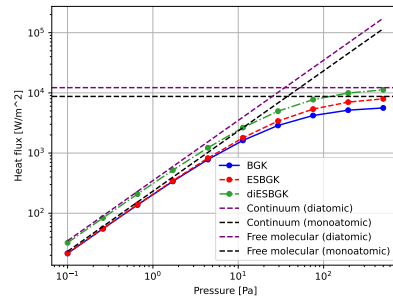
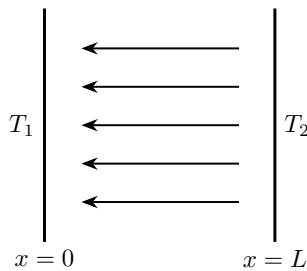


Figure 1: A schematic of the Fourier flow test case (left): hydrogen gas between two parallel walls with temperatures $T_1 < T_2$. The heat flux as a function of the pressure of the gas for several BGK models (right).

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Evolutionary PDEs for Macroscopic Modeling of Road Traffic Flow in Urban Areas

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Keywords: macroscopic traffic model; Vlasov-Poisson equation; numerical methods; experimental validation method.

ABSTRACT

Modeling traffic flows on complex road networks in urban agglomerations is a pressing issue. Populations in large cities are growing, and the level of motorization is correspondingly rising. Traditionally, transport trajectories are described by motion equations in the form of ordinary differential equations for the coordinates and velocity of an object. It was noted [1] that such equations of motion are characteristics of the Vlasov equation. Transport move based on the driver's decisions based on information coming from outside. In the field of information transmission, the Friis transmission equation can be used, which defines the signal decrease as the square of the distance between objects. This approach allows us to use the Poisson equation to obtain the potential of the information field. The report presents a new dynamic model of traffic flows based on solving the system of Vlasov-Poisson equations. The equations of motion include a function that ensures the absence of traffic collisions and the intensity of the information field. The Poisson equation right-hand side depends on the density of the traffic flow. The solution of the Vlasov-Poisson equation system can be implemented by the finite-difference method and the PIC method. The model takes into account local and long-range interactions, and scales well. The results of computational experiments for basic road scenarios are presented. We have also developed software for video analysis of traffic, which allows us to obtain real characteristics of the traffic flow to verify our model.

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Anisotropic mesh adaptation for the non-linear shallow water equations

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Keywords: discontinuous Galerkin finite element schemes; anisotropic mesh adaptation;

ABSTRACT

Accurate forecasting of flooding from natural disasters such as tsunamis and storms can be crucial for preventing catastrophic damage to lives and buildings. One avenue to do so is through the use of numerical methods, i.e. computer simulations of such events by modeling fluid motion via equations such as the non-linear shallow water equations. In order to numerically solve such equations, a mesh which subdivides the domain into a conformal tessellation is often used. The distribution and shape of the elements which comprise the mesh can be adapted based on various methods to distribute computational resources such that areas of higher complexity are given more resolution. One such method is metric-based anisotropic mesh adaptation, which is used in this work to demonstrate its suitability for predicting coastal run-up from incident waves. This method has not been used for this application in the literature, and a unique choice of error indicator as well as a simple wet-dry interface treatment for solution transfer is presented.

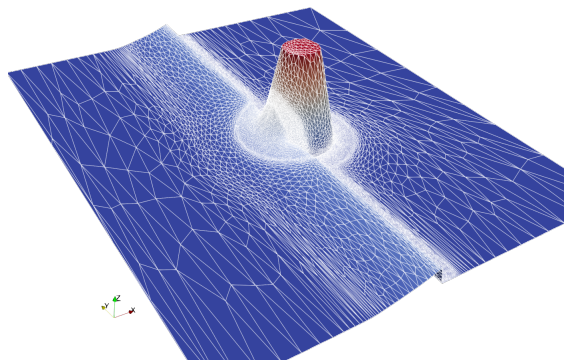


Figure 1: Adapted mesh, conical island test case, $t = 5.6$

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A structure preserving finite element scheme for the incompressible GPR model

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Keywords: continuum mechanics; structure-preserving; curl-free, finite volume schemes; compatible finite elements; Eulerian methods; high order methods.

ABSTRACT

The Godunov–Peshkov–Romenski (GPR) model [1, 2] provides a unified continuum-mechanics framework for the description of both solids and fluids. In this presentation, I introduce a novel finite element discretization for the incompressible GPR model that preserves the underlying geometric structure of the equations. The proposed method is pressure-robust and conserves the curl of the distortion field in the solid regime. Time discretization is achieved through a splitting strategy combined with the average vector field method [3], yielding exact energy preservation or dissipation at the discrete level. I further discuss possible upwind and stabilization techniques and analyze their interaction with the structure-preserving properties of the scheme. While many ideas can be adapted from incompressible magnetohydrodynamics, which exhibits a closely related mathematical structure, the incompressible GPR model introduces additional challenges, such as the non-quadratic nature of its energy functional. The presentation concludes with numerical experiments conducted using the open-source finite element library NGSolve [4].

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High order numerical schemes within a new first-order BSSNOK formulation of the Einstein equations

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ABSTRACT

I will present some recent progresses in the numerical solution of the Einstein equations which combine a new first order (in space and time derivatives) formulation of the BSSNOK Einstein–Euler system and high order numerical schemes. In particular, I will discuss central WENO (conservative) finite difference and ADER-Discontinuous Galerkin (DG) schemes. Well Balancing is an extra feature for the treatment of (close to-) equilibrium configurations while a subcell limiter is adopted for handling discontinuities in the DG discretization. This approach allowed to compute the first gravitational wave signal emitted by the binary merger of two black holes and obtained with a first order formulation of the BSSNOK equations. In addition, I will describe preliminary steps to derive the Einstein equations as a rigorous Euler-Lagrange system. Having the goal to obtain a non-linearly stable scheme for numerical relativity, I will review the attempts that could make this result possible, starting from Einstein choice, following Landau arguments and highlighting the main difficulties.

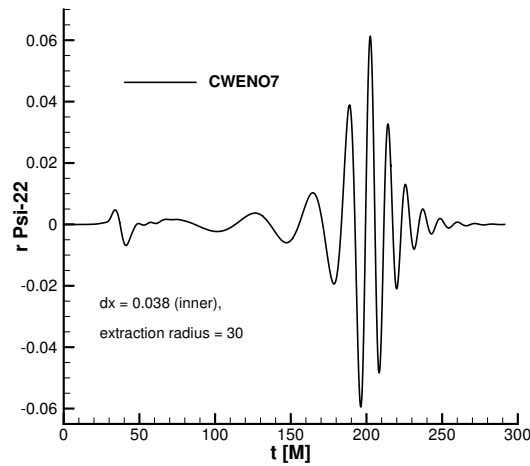


Figure 1: Gravitational waveform computed with a first order formulation of the BSSNOK Einstein equations.

Extending Quinpi in 2D: an implicit scheme for nonlinear conservation laws

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Keywords: finite volume schemes; implicit methods; high order methods; time limiting; numerical entropy production.

ABSTRACT

When dealing with stiff conservation laws, explicit time integration forces to employ very small time steps, due to the restrictive CFL stability condition. Implicit methods offer an alternative, yielding the possibility to choose the time step according to accuracy constraints. However, the construction of high-order implicit methods is difficult, mainly because of the non-linearity of the space and time limiting procedures required to control spurious oscillations. The Quinpi approach addresses this problem by introducing a first-order implicit predictor, which is employed in both space and time limiting. The scheme has been proposed in (Puppo et al., *Comm. Comput. Phys.*, 2024) for systems of conservation laws in one dimension. In this talk, I will present the multi-dimensional extension. Similarly to the one-dimensional case, the scheme combines a third-order Central WENO-Z reconstruction in space with a third-order Diagonally Implicit Runge-Kutta (DIRK) method for time integration, and a low order predictor to ease the computation of the Runge-Kutta stages. Even applying space-limiting, spurious oscillations may still appear in implicit integration, especially for large time steps. For this reason, a time-limiting procedure inspired by the MOOD technique and based on numerical entropy production together with a cascade of schemes of decreasing order is applied. The scheme is tested on the Euler equations of gasdynamics, also in low Mach regimes. The numerical tests are performed on both structured and unstructured meshes.

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A priori time limiting for the Quinpi scheme

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Keywords: implicit methods; CWENO approximation; finite volume schemes; hyperbolic systems.

ABSTRACT

Stiff hyperbolic problems can be challenging for numerical methods to solve. The Quinpi numerical method [2, 3] uses an implicit third-order Central Weighted Essentially Non-Oscillatory (CWENO) space reconstruction and a third-order Diagonally Implicit Runge-Kutta (DIRK3) scheme for the time integration. Despite limiting the solution in space, spurious oscillations can occur at large time steps. Therefore, in this work, an a priori time-limiting technique, adopted from [1], is presented. The oscillations are reduced using a weighting procedure inspired by spatial WENO methods, where the weights combine the high-order Runge-Kutta scheme with the composite Backward Euler scheme. The properties of the solution from each stage are investigated. Since the solutions obtained in the first two stages of DIRK3 inherit strong stability properties, we aim to limit only in the last stage of the DIRK3 scheme. Therefore, for the smoothness indicator, flux differences from the previous stages and from the previous time step are used together with the corresponding coefficients of the Butcher tableau of the DIRK3 scheme. The a priori technique is applied and compared on the scalar linear and nonlinear conservation laws, and also on the linear and nonlinear systems of equations. The point of interest is the system of the Euler equations in the low-Mach regime.

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